Coordinates Revision Guide

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Contents

1 Introduction

Throughout your time doing Physics at Warwick you will find that you will need to convert between different coordinate systems. Hopefully this guide will capture all the necessary conversions of mathematical entities such as differential operators, integral elements etc, for different coordinate systems. In particular we will be focusing on transforming between Cartesian, Cylindrical and Spherical coordinates since these are the only coordinate systems we really deal with in this course.

1.1 Some motivational words

This guide is intended for all Physics/Maths and Physics students of all year groups however I want to slightly diverge into a brief discussion and somewhat motivational few words for students, in particular those whom are in their first year. The transition from A-levels into university may seem to be more difficult than the transition from GCSE to A-level, however do not be discouraged by this jump in difficulty. Everybody is in the same boat of finding the content difficult, it is through consistency of keeping up with the weekly problem sheets, attending all (well, maybe most) of your lectures and keeping up with the module notes that you will achieve the grade you wish for. It is more efficient for you to study in smaller quantities over a long period of time as opposed to cramming it all last minute a few weeks/days before your exams. Aside from this, we hope this guide makes the conversion between different coordinate systems easier to understand and enjoy your time at Warwick as much as you can.

2 Transforming Coordinate Systems

2.1 Cartesian Coordinates

Cartesian coordinates are the generic x, y, z coordinates that we should be familiar with. They form an orthogonal set of 3 planes, and a point in Cartesian coordinates is given by a vector, $\vec{\boldsymbol{r}} = (x, y, z)$, where x, y and z can take any real value. Another way of writing this vector is in terms of the basis vectors, $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}}$ or $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. For the purpose of this guide I will stick to using $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ since I find the other notation cringe.

The basis vectors, $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ have euclidean length of one, and point in the x, y and z directions respectively. Thus, taking the scalar (dot) products of them gives the useful results:

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1$$

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}} = 0$$

where we remind the reader the definition of the scalar (dot) product:

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \cos\theta$$

such that θ is the angle between the two vectors.

2.2 Cylindrical Polar Coordinates

Cylindrical polar coordinates can be used to exploit cylindrical symmetry. A common example is in Electricity & Magnetism where you might be required to calculate the electric field of a cylindrical wire element. Performing integrals for this sort of problem in Cartesian coordinates gets very messy very quickly, so by using cylindrical polars, problems can be made easier.

With these coordinates, we go from $\vec{\boldsymbol{r}} = (x, y, z)$ to $\vec{\boldsymbol{r}} = (\rho, \varphi, z)$. Here, ρ is the length of the x-y plane projection, or more concisely, the radial distance. This has a range of $(0 \leq \rho < \infty)$. φ is the azimuthal angle which is the angle from the x-axis in the x-y plane, with a range of $(0 \leq \varphi < 2\pi)$ and z is the axial direction, with a range of $(-\infty < z < \infty)$, same as Cartesian coordinates.

Remark: Sometimes the azimuthal angle will be described to be in the range $(-\pi \leq \varphi < \pi)$

Cylindrical polars in terms of Cartesian coordinates are as follows:

$$\rho = \sqrt{x^2 + y^2}$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

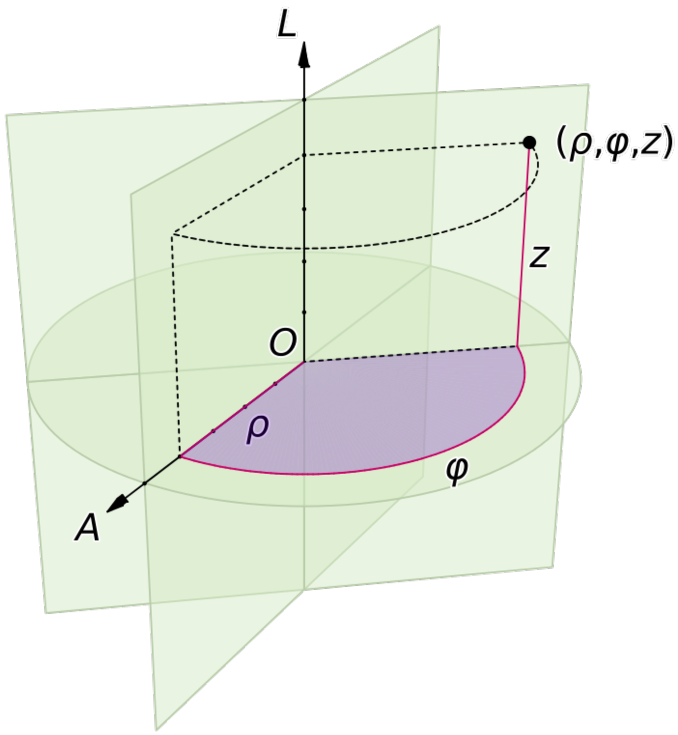

Figure 1: An illustration of the cylindrical coordinate system.

$$z = z$$

Cartesian coordinates in terms of cylindrical polars are as follows:

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

The basis vectors in cylindrical coordinates are different to those in Cartesian coordinates. In terms of the Cartesian basis vectors, they are as follows:

$$\hat{\boldsymbol{\rho}} = \cos \varphi \hat{\boldsymbol{x}} + \sin \varphi \hat{\boldsymbol{y}} \equiv \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{\sqrt{x^2 + y^2}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin \varphi \hat{\boldsymbol{x}} + \cos \varphi \hat{\boldsymbol{y}} \equiv \frac{-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}}{\sqrt{x^2 + y^2}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

Then, it's possible to define the Cartesian basis vectors in terms of the cylindrical basis vectors as follows:

$$\hat{\boldsymbol{x}} = \cos \varphi \hat{\boldsymbol{\rho}} - \sin \varphi \hat{\boldsymbol{\varphi}} \equiv \frac{x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}}$$
$$\hat{\boldsymbol{y}} = \sin \varphi \hat{\boldsymbol{\rho}} + \cos \varphi \hat{\boldsymbol{\varphi}} \equiv \frac{y\hat{\boldsymbol{\rho}} + x\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

Finally, it's useful to know the time derivatives of the basis vectors, since they vary in time differently than Cartesian basis vectors.

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \frac{d\varphi}{dt}\hat{\boldsymbol{\varphi}} = \dot{\varphi}\hat{\boldsymbol{\varphi}}$$

$$\frac{d\hat{\boldsymbol{\varphi}}}{dt} = -\frac{d\varphi}{dt}\hat{\boldsymbol{\rho}} = -\dot{\varphi}\hat{\boldsymbol{\rho}}$$
$$\frac{d\hat{\boldsymbol{z}}}{dt} = 0$$

2.3 Spherical Polar Coordinates

Spherical polar coordinates are really useful in physics. They crop up in quantum mechanics and electricity & magnetism a lot, and as with cylindrical coordinates, using them to exploit spherical symmetry makes calculations much easier.

Here, we go from $\vec{\boldsymbol{r}} = (x, y, z)$ to $\vec{\boldsymbol{r}} = (r, \theta, \phi)$. r is the radial distance, ranging from $(0 \leq r < \infty)$, θ is the polar angle between the z-axis and the radial vector connecting the origin to the point in question, ranging from $(0 \leq \theta \leq \pi)$ and as with cylindrical polars, ϕ is the azimuthal angle defined to be the angle between the x-axis and the projection of the radial vector onto the x-y plane. ranging from $(0 \leq \phi < 2\pi)$. Spherical polar coordinates in terms of Cartesian coordinates can be represented as follows:

Remark: Some literature swap the definitions of ϕ and θ so be wary of this when reading other sources.

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \arccos\left(\frac{z}{r}\right)$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$

Then, Cartesian coordinates are given in terms of spherical polars:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

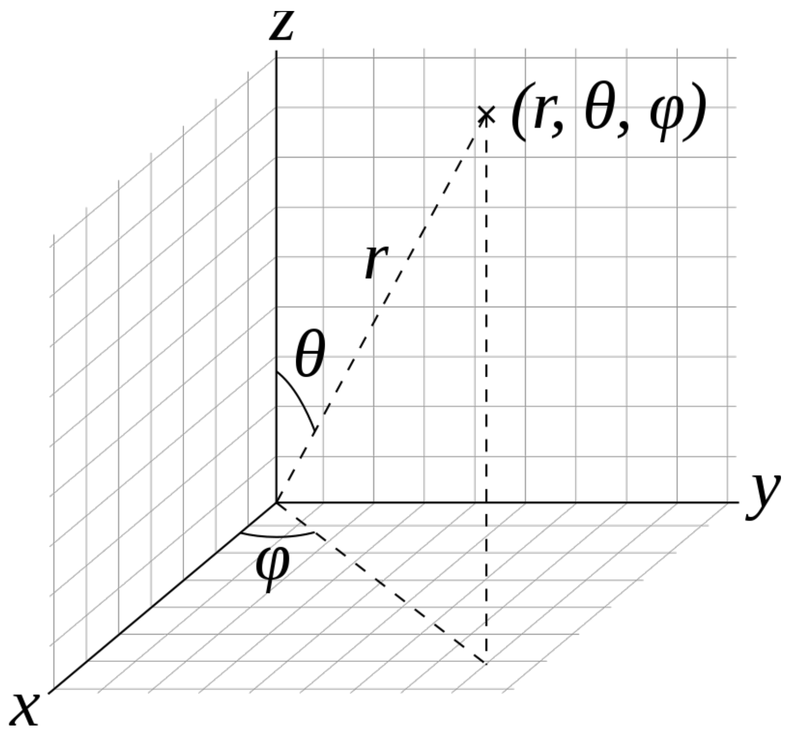

Figure 2: An illustration of the spherical coordinate system.

The spherical basis vectors are also different. In terms of the Cartesian basis vectors, they are as follows:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\cos\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$$

The Cartesian basis vectors in terms of spherical polar basis vectors can be defined too:

$$\hat{\boldsymbol{x}} = \sin\theta\sin\phi\hat{\boldsymbol{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{y}} = \sin\theta\sin\phi\hat{\boldsymbol{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{z}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

Again, it's useful to know the time derivatives of the basis vectors, just in case:

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}$$
$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\boldsymbol{r}} + \dot{\phi}\cos\theta\hat{\boldsymbol{\phi}}$$
$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\sin\theta\hat{\boldsymbol{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}$$

2.4 Hyperbolic Coordinates

This is a relatively new second year topic, but it is quite straightforward compared to the previous coordinate systems! First of all, if you're reading this in your first year, don't worry about this section. Secondly, these coordinates are used in 2D only, at least for the PX275 course.

It's probably best to introduce this system with an example; consider the thermodynamics of an ideal gas. We'll use the ideal gas equation, $pV = nRT$.

Here, p is the pressure, V is the volume, n is the number of moles, R is the molar gas constant, equal to 8.31 J mol^{-1} K^{-1}, and T is the absolute temperature of the gas. We often want to plot pV graphs - pressure against volume. However, with the ideal gas equation, we see that we'll end up with a hyperbolic line, i.e. $p \propto V^{-1}$. See Fig. 3 for an example of the plot.

So, from this graph, you might be asked to find the area enclosed between the 4 lines. This would be a bit of a nightmare to integrate normally, so, switching coordinate systems can make this a lot easier. The transforms needed are:

$$pV = r$$
$$\frac{p}{V} = s$$

Let's plot these with the same values used for Fig. 3: So, our integral becomes a lot easier, however, we do need to find the Jacobian determinant to perform it. The integral becomes:

$$\Omega = \int_r \int_s |\boldsymbol{J}| dr ds$$

Where Ω is the area enclosed by the 4 lines, and $|\boldsymbol{J}|$ is the Jacobian determinant. The Jacobian matrix for hyperbolic coordinates is as follows:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial V}{\partial r} & \frac{\partial V}{\partial s} \\ \frac{\partial p}{\partial r} & \frac{\partial p}{\partial s} \end{bmatrix}$$

That's all that's taught in the lecture notes. It's worth mentioning that these coordinates can be used with other equations, such as $V = IR$, or $c = f\lambda$. Essentially, if you have an equation of the form $xy = c$, where c is a constant, you'll end up with hyperbolic lines.

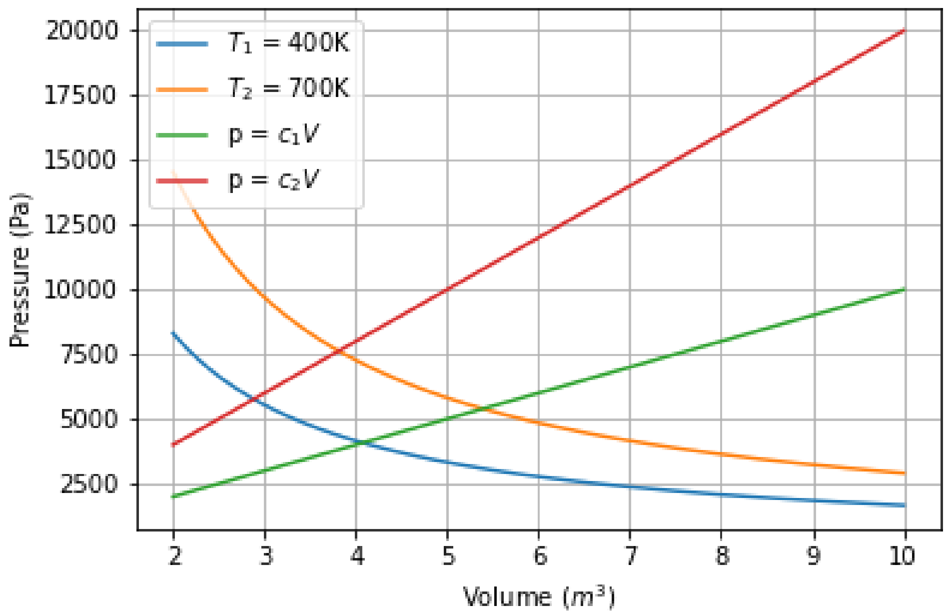

Figure 3: Hyperbolics for the ideal gas plotted. The red and green lines show $p = c_1V$ and $p = c_2V$, i.e. two different heat capacities. The orange and blue lines show $p = \frac{nRT}{V}$ at two temperatures.

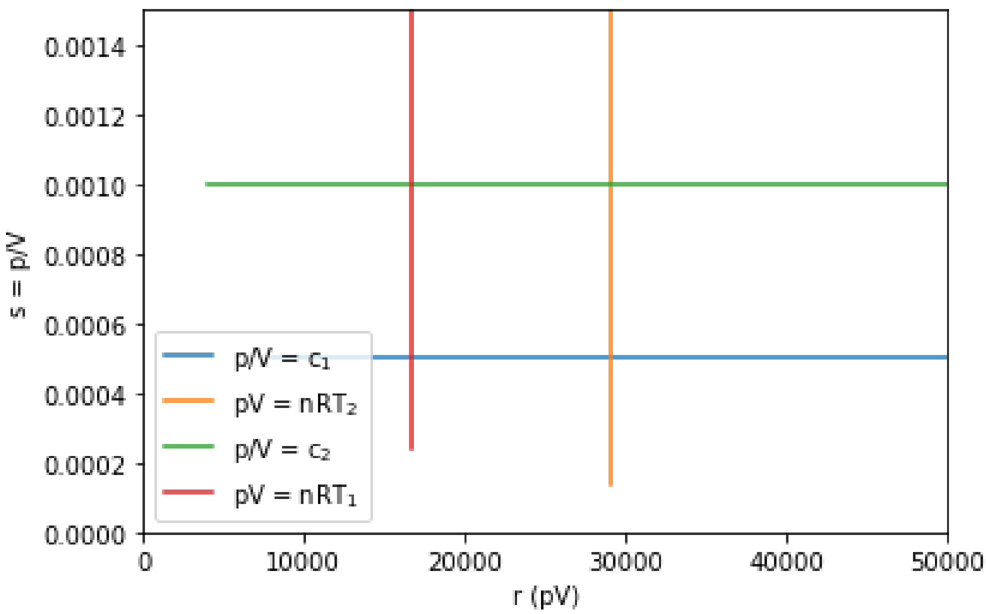

Figure 4: Transformed coordinates for the ideal gas. Along the blue line, p/V is constant, and along the orange line, pV is constant as well, so we have straight lines, i.e. a rectangular area, which is much easier to integrate over.

3 Area and Volume elements for integration

3.1 Cartesian coordinates

Definition: The differential area element (dA) and differential volume element (dV) are defined in Cartesian coordinates:

$$dA = \begin{cases} dydz & \text{along the } \hat{\boldsymbol{x}} \text{ direction} \\ dxdz & \text{along the } \hat{\boldsymbol{y}} \text{ direction} \\ dxdy & \text{along the } \hat{\boldsymbol{z}} \text{ direction} \end{cases}$$

$$dV = dxdydz$$

3.2 Cylindrical polar coordinates

Definition: The differential area element (dA) and differential volume element (dV) are defined in cylindrical polar coordinates:

$$dA = \begin{cases} \rho \, d\varphi dz & \text{along the } \hat{\boldsymbol{\rho}} \text{ direction} \\ d\rho dz & \text{along the } \hat{\boldsymbol{\varphi}} \text{ direction} \\ \rho \, d\rho d\varphi & \text{along the } \hat{\boldsymbol{z}} \text{ direction} \end{cases}$$

$$dV = \rho \, d\rho d\varphi dz$$

3.3 Spherical polar coordinates

Definition: The differential area element (dA) and differential volume element (dV) are defined in spherical polar coordinates:

$$dA = \begin{cases} r^2 \sin\theta \, d\theta d\phi & \text{along the } \hat{\boldsymbol{r}} \text{ direction} \\ r \sin\theta \, drd\phi & \text{along the } \hat{\boldsymbol{\theta}} \text{ direction} \\ r \, drd\theta & \text{along the } \hat{\boldsymbol{\phi}} \text{ direction} \end{cases}$$

$$dV = r^2 \sin\theta \, drd\theta d\phi$$

3.4 Using these elements

These elements can be derived either geometrically, or by using the Jacobian matrix determinant, which is a bit beyond the scope of this guide. However, using these elements is relatively simple, which requires a basic understanding of integration and multiple integrals.

Say for example, you wanted to calculate the volume of sphere. We know the formula already, $V = \frac{4}{3}\pi r^3$. However, we can set up a triple integral in spherical coordinates to verify this result.

Let's assume we have a sphere of radius a, centred on the origin. We can calculate the volume using a triple integral:

$$V = \iiint_V dV$$

Then, we know our volume element from section 4.3, $dV = r^2 \sin\theta \, drd\theta d\phi$. For our sphere, we know that r ranges from 0 to a (the radius of the sphere given to us), the polar angle must range from 0 to π and the azimuthal angle must range from 0 to 2π. Therefore, these are our limits of integration. Let's set up our integral:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin\theta \, drd\theta d\phi$$

r and θ don't depend on ϕ, so we can just take the factor of 2π outside the integral, giving us:

$$V = 2\pi \int_0^{\pi} \int_0^{r} r^2 \sin\theta \, drd\theta$$

Then, integrating the radial part gives us:

$$V = 2\pi \int_0^{\pi} \int_0^{a} r^2 \sin\theta \, drd\theta = 2\pi \int_0^{\pi} \left[\frac{1}{3} r^3\right]_0^a \sin\theta \, d\theta$$

This leaves us with:

$$V = \frac{2}{3}\pi a^3 \int_0^{\pi} \sin\theta \; d\theta$$

This is a straightforward integral:

$$\int_0^{\pi} \sin\theta \; d\theta = [-\cos\theta]_0^{\pi} = 1 + 1 = 2$$

Finally, combining all of this gives:

$$V = \frac{2}{3}\pi a^3 2 = \frac{4}{3}\pi a^3$$

Which is the result we expect. This is a very trivial example, however it sets a foundation for trickier problems, like calculating volumes of semispheres or sections of spheres, which is done by adjusting the limits of integration carefully. A useful tip will be to sketch the geometry of the problem.

4 Del Operator

4.1 Preliminary assumptions

Just a brief note that we are assuming doubly continuous differentiability for any arbitrary scalar function f and vector field $\vec{\boldsymbol{A}}$. This just essentially means that f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial z^2}$ are all continuous functions with no discontinuity on the whole real line and similarly for the components of $\vec{\boldsymbol{A}}$.

4.2 Cartesian Coordinates

Definition: Given some scalar function $f = f(x, y, z)$ and a vector field $\vec{\boldsymbol{A}} = (A_x, A_y, A_z)$, in Cartesian coordinates, we define the del (sometimes called the grad or nabla) operator as:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Thus we define the following operations: the gradient and Laplacian of a scalar function f, as well as the divergence and curl of a vector field, $\vec{\boldsymbol{A}}$, with respect to Cartesian coordinates is defined as follows:

$$\text{grad}(f) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\text{div}(\vec{\boldsymbol{A}}) = \nabla \cdot \vec{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl}(\vec{\boldsymbol{A}}) = \nabla \times \vec{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta \vec{\boldsymbol{A}} = \nabla^2 \vec{\boldsymbol{A}} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right)$$

Remark: One should clarify that the gradient operator can only be applied to a scalar function f whereas the divergence and curl operators can only apply to vector fields $\vec{\boldsymbol{A}}$. However, the Laplacian can operate on both vector and scalar fields.

Remark: We must remember that grad(f), curl$(\vec{\boldsymbol{A}})$ and $\nabla^2 \vec{\boldsymbol{A}}$ return a vector giving us essentially three equations to work with in contrast to div$(\vec{\boldsymbol{A}})$ and Δf which compute a scalar quantity.

4.3 Cylindrical Coordinates

Similar to the case for Cartesian coordinates we define a scalar function $f = f(\rho, \varphi, z)$ and a vector field $\vec{\boldsymbol{A}} = (A_\rho, A_\varphi, A_z)$ taking note that now these are defined with respect to the cylindrical coordinate system. Again, let's define the del operator:

$$\nabla = \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\right)$$

Now we define the grad(f), div($\vec{\boldsymbol{A}}$), curl($\vec{\boldsymbol{A}}$) and Laplacian operator with respect to Cylindrical polar coordinates:

$$\nabla f = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial x} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{\boldsymbol{A}} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}, \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}, \frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right)$$

$$\Delta f = \nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

Remark: the operators in cylindrical coordinates are in a more complex form than those seen for Cartesian coordinates, it is imperative that you do not miss out any terms when computing these.
Remark: most likely you will not have to remember these, they will be given to you in the rubric of the exam.

4.4 Spherical Coordinates

Setting things up, we define the scalar function $f = f(r, \theta, \phi)$ and the vector field $\vec{\boldsymbol{A}} = (A_r, A_\theta, A_\phi)$. We define the del operator:

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

Defining the same operators as before in terms of spherical coordinates:

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{\boldsymbol{A}} = \left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right), \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right), \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\right)$$

$$\Delta f = \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Remark: Thankfully, you will not have to in general use any more than grad(f) or curl($\vec{\boldsymbol{A}}$) in spherical polar coordinates.

It should also be noted that the following useful vector identities come up frequently:

$$\nabla \cdot \nabla f = \nabla^2 f = \Delta f$$

$$\nabla \times \nabla f = \vec{0}$$

$$\nabla \cdot (\nabla \times \vec{\boldsymbol{A}}) = 0$$

$$\nabla \times (\nabla \vec{\boldsymbol{A}}) = \nabla(\nabla \cdot \vec{\boldsymbol{A}}) - \nabla^2 \vec{\boldsymbol{A}}$$

$$\nabla \cdot (f\vec{\boldsymbol{A}}) = \vec{\boldsymbol{A}} \cdot \nabla f + f(\nabla \cdot \vec{\boldsymbol{A}})$$

5 Final remarks

We hope that you have found this revision guide useful and we hope to create many more in the upcoming academic year. It is worth mentioning that there are many resources online such as Wikipedia and YouTube that will help you understand this topic in great depth. I feel it is worth mentioning a particular website, namely: https://tutorial.math.lamar.edu/ which is an excellent website for revision on mathematical topics such as the ones seen today (I should be sponsored by them).