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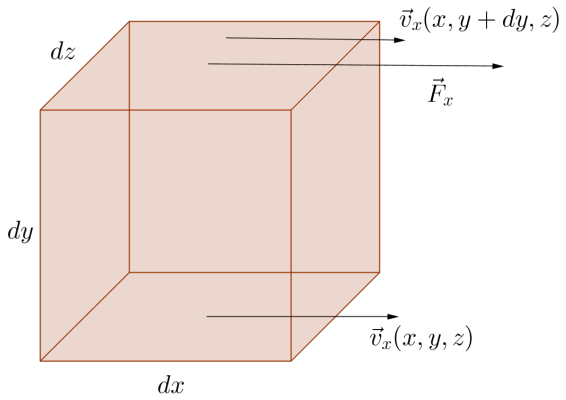
1 Foundations

1.1 Fluids vs Solids

Fluids can be described as an extreme of solid matter, which does not accomodate shear stress. This means that no matter the shear deformation caused, there is no response force to a shear stress acting on a liquid. Imagine a volume of the fluid dV with sides dx, dy and dz, as in figure below.



In solids, we would write that for final state (when the forces are balanced)

$$\tau_x = Ge_x$$

where $\tau = \frac{F_x}{dxdz}$ is the shear stress, e_x is the shear strain (both in x direction) and G is some constants. We however, cannot write this in the case of fluids. But, for so called Newtonian fluids

$$\tau_x = \mu \frac{de_x}{dt}$$

where μ is a constant, called the viscosity.
Therefore, for these fluids, a uniform stress creates a uniformly increasing strain.
The shear strain can be connected to the velocity of the fluid as follows. Let the base of our volume move at speed $v_x(x, y, z)$ and the top at speed $v_x(x, y + dy, z)$. By Taylor expansion

$$v_x(x, y + dy, z) \approx v_x(x, y, z) + \frac{\partial v_x}{\partial y} dy$$

Hence, the distance traveled by the top relative to the bottom in time dt is

$$ds = (v(x, y + dy, z) - v(x, y, z))dt = \frac{\partial v_x}{\partial y} dydt$$

But, this distance has to be equal to the distance travelled due to shear strain. Shear strain is relative, and hence the change of shear strain over time dt is

$$de_x = \frac{ds}{dy} = \frac{\partial v_x}{\partial y} dt$$

Therefore, we have

$$\frac{de_x}{dt} = \frac{\partial v_x}{\partial y} \tag{1}$$

Therefore, we have for Newtonian fluids

$$\tau_x = \mu \frac{\partial v_x}{\partial y} \tag{2}$$

For non-Newtonian fluids, we would have some different dependence of stress and change in strain - for example, for very fast changing stress, the strain might not change very much - the viscosity would be very high for quickly changing stress. The precise description of such fluids is quite hard and is not considered in this module.

1.2 Fluid Motion

There are two distant approaches how to imagine the motion in the fluid. First, so called Lagrangian approach, is to try to model small elements of the fluid travelling in some space and interacting with each other. This method is rarely used as the number of degrees of freedom is usually very high and trajectories of all separate particles need to be tracked through all space.
Other approach, also called Euler approach, is to imagine a small immobile volume in the fluid, which we observe. We can then describe at which speed does the fluid enter the volume, what is the difference of forces acting on a fluid at the boundaries of this control volume, etc. This approach is usually taken, as the basic vector field, which is the fluid velocity field, is fairly easy to understand in this picture. Also, the fact that this control volume is stationary makes geometric intuition easier in this picture.

1.3 Poiseuille Flow

1.3.1 Two Dimensions

Imagine a two dimensional tube as a rectangle of length L along the x axis and radius r along the y axis. Now assume that the pressure gradient in the fluid over the length of the pipe is Q, where

$$Q = \frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L}$$

where p_1 is the pressure at the beginning of the tube ($x = 0, y = 0$) and p_2 is the pressure at the end of the tube($x = L, y = 0$). Now imagine a small control element dS. Let viscous forces acting on this element in the direction of x are $F_v(x, y + dy)$ and $F_v(x, y)$. The corresponding stresses are $\tau_v(x, y + dy) = \frac{F_v(x,y+dy)}{dx}$ and $\tau(x, y) = \frac{F_v(x,y)}{dx}$. These stresses cause a strain on top and bottom of the control volume at the same time. The overall strain of the top is then only the relative strain. Let the strain at the top over time dt be de_t and the corresponding strain at the bottom be de_b. The relative strain of the top is then

$$de = de_t - de_b = \frac{de}{dt}\bigg|_{x=x,y=y+dy} dt - \frac{de}{dt}\bigg|_{x=x,y=y} dt$$

Using the relation (1) (and dropping the explicit $x = x$)

$$de = \frac{\partial v_x}{\partial y}\bigg|_{y=y+dy} dt - \frac{\partial v_x}{\partial y}\bigg|_{y=y} dt$$

$$\frac{de}{dt} = \frac{\partial v_x}{\partial y}\bigg|_{y+dy} - \frac{\partial v_x}{\partial y}\bigg| y = y$$

Multiplying both sides by μdx, we see that the equation becomes force equation

$$F_{vt} = \mu\frac{de}{dt}dx = \mu dx\frac{\partial v_x}{\partial y}\bigg|_{y=y+dy} - \mu dx\frac{\partial v_x}{\partial y}\bigg|_{y=y} = F_v(x, y + dy) - F_v(x, y)$$

Using the Taylor expansion for the second expression

$$F_{vt} = \mu\frac{\partial^2 v_x}{\partial y^2}dxdy$$

In a stable flow, this force has to be balanced by some other force. In our case, this is the pressure force. The total pressure force is given as (in positive x direction)

$$F_{pt} = F_p(x, y) - F_p(x + dx, y) = p(x, y)dy - p(x + dx, y)dy = -\frac{\partial p}{\partial x}dxdy = -Qdxdy$$

If these forces are balanced

$$F_{vt} + F_{pt} = 0$$

$$-F_{pt} = F_{vt}$$

And thus

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 v_x}{\partial y^2} \tag{3}$$

For our constant pressure gradient, we then have

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{Q}{\mu}$$

$$v_x = \frac{1}{2\mu} Q y^2 + ay + b$$

Where a and b are some constants. But, from an experimental setup, we know that the velocity of the fluid near the boundary with solid tends to zero. Thus

$$v_x(x, r) = v_x(x, -r) = 0$$

From the first equation, we have

$$\frac{1}{2\mu} Q r^2 + ar + b = 0$$

$$b = -\frac{1}{2\mu} Q r^2 - ar$$

From the second equation, we have

$$\frac{1}{2\mu} Q r^2 - ar + b = 0$$

$$-ar - ar = 0$$

$$a = 0$$

Therefore

$$b = -\frac{1}{2\mu} Q r^2$$

And therefore, substituting for Q

$$v_x(x, y) = \frac{p_1 - p_2}{2\mu L}(r^2 - y^2) \tag{4}$$

Hence, the velocity profile is parabolic.

1.3.2 Three Dimensions

In tree dimensions, the setup is similar - we have a cylindrical tube of constant radius R in cylindrical polar coordinates (r, ϕ, z), with z axis coinciding with the axis of the tube. The relative strain force on an element of volume $dV = r d\phi dr dz$ in the z direction is

$$F_{vt} = \mu \frac{\partial v_z}{\partial r}\bigg|_{r=r+dr} (r + dr) d\phi dz - \mu \frac{\partial v_z}{\partial r}\bigg|_{r=r} r d\phi dz = \mu \frac{\partial^2 v_z}{\partial r^2} r dr d\phi dz + \mu \frac{\partial v_z}{\partial r} dr d\phi dz =$$

$$= \mu \left(r \frac{\partial^2 v_z}{\partial r^2} + \frac{\partial v_z}{\partial r} \right) \frac{dV}{r} = \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) dV$$

The pressure force is

$$F_{pt} = (p(r, \phi, z) - p(r, \phi, z + dz)) r d\phi dr = -\frac{\partial p}{\partial z} r dr d\phi dz = -\frac{\partial p}{\partial z} dV$$

Hence, at the balance

$$\frac{\partial p}{\partial z} = \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \tag{5}$$

Again, assuming linear pressure gradient $Q = \frac{p_2 - p_1}{L}$

$$\frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = \frac{Q}{\mu} r$$

$$r \frac{\partial v_z}{\partial r} = \frac{Q}{2\mu} r^2 + a$$

$$\frac{\partial v_z}{\partial r} = \frac{Q}{2\mu} r + \frac{a}{r}$$

$$v_z = \frac{Q}{4\mu}r^2 + a\ln r + b$$

where a and b are some constants. In order for v_z to be finite at $r = 0$ (a physical requirement), we need $a = 0$. From the boundary condition then

$$v_z(r = R, \phi, z) = 0$$

$$\frac{Q}{4\mu}R^2 + b = 0$$

$$b = -\frac{Q}{4\mu}R^2$$

And thus

$$v_z(r) = \frac{p_1 - p_2}{4\mu L}(R^2 - r^2) \tag{6}$$

1.4 Viscosity

We have now done a few calculations with viscosity, but have not discussed its origin. Basically, all interaction inside the medium can occure at speed up to the speed of the sound in the medium. In liquid, the speed of the sound is of the order of speed of separate molecules, which is usually big, but not infinite. Therefore, as force is applied to the fluid, only some molecules can transfer the momentum away from the force applied.
Usually, for liquids, as temperature increases, the speed of the molecules increases and thus the maximum transfered momentum increases, and therefore the interaction speed increases and viscosity decreases (more strain per the same stress).
On the other hand, in gases, with increasing temperature the volume usually increases more than the speed increase can compensate for, and so the viscosity increases overall.

1.5 Reynolds Number

Usually, the flow of the fluid can become turbulent, meaning that the velocity field is no longer stationary in time. To try to determine when does this transition take place, we try to use dimensional analysis to find some dimensionless number that would help us determine the transition.
The variables considered should be the density of the fluid ρ, the average speed of the fluid v, average size of the space perpendicular to the speed direction d (for pipe of radius r, $d = 2r$), viscosity of the fluid μ, and perhaps also the pressure gradient Q.
But, just from the previous four variables, we can create a dimensionless ratio

$$Re = \frac{\rho v d}{\mu}$$

This ratio is called the Reynolds number. We would expect that for very fast and not very viscous fluids, the flow will be turbulent. This corresponds to a very high Reynolds number ($Re >> 1$). On the other hand, small Reynolds number implies that the flow is probably laminar and stable in time.

1.6 Visualising Flow

The flow is usually represented by a lines that somehow correspond to the velocity field of the flow. Few examples are now listed

1.6.1 Streamlines

Streamlines are lines that are at every point tangential to the velocity field. In 2D, this requires that the streamline $y(x)$ follows

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

Because at every point streamlines can be used to determine the direction of v, for region with well defined v, the streamlines cannot intersect.

1.6.2 Pathlines

Pathlines follow certain particle in its flow through some volume. In laminar flow, these are identical to streamlines, but in turbulent flow, they are generally different.

1.6.3 Streaklines

Streaklines are lines created by particles emerging just from some controlled area/volume. We can imagine creating them by adding some drop of colouring to the fluid and tracing its path throught the fluid. Again, they are identical with streamlines in the limit of steady flow.

1.6.4 Streamtubes

Now, if somewhere in the flow there is a tube that is created by streamlines with identical direction, the particles inside this tube cannot escape it, as this would cause the streamlines to cross, which does not occur. Therefore, these tubes act as real tubes, even though they are only created by imaginary streamlines. These tubes are called streamtubes and the fact that they form closed systems can be used for predictions about the fluid.

2 Bernoulli Equation

Bernoulli equation is the energy conservation statement in the context of fluids. Imagine a part of a streamtube somewhere in the flow, with beginning area A_1 and velocity v_1 and ending area A_2 and velocity v_2. The mass entering the tube is $v_1 A_1 \rho_1$, which must be the same as mass exiting the tube, so

$$v_1 A_1 \rho_1 = v_2 A_2 \rho_2$$

Now imagine a small time dt passes. This means that the mass

$$dm = v_1 A_1 \rho_1 dt$$

enters the tube. The kinetic energy of this mass is

$$dE_k = \frac{1}{2} dm v_1^2$$

The internal energy of the mass is

$$dU = u_1 dm$$

where u is the internal energy mass density. The gravitational energy of the mass is

$$dV = dmgh_1$$

where g is the acceleration due to gravity and h is the distance above the zero potential energy level. Hence, the total energy of the mass is

$$dE_1 = dm\left(\frac{1}{2}v_1^2 + u_1 + gh_1\right)$$

Similarly for the emergent mass

$$dE_2 = dm\left(\frac{1}{2}v_2^2 + u_2 + gh_2\right)$$

The difference of energies between these two points in the flow must be equal to the power energy supplied by the pressure from the ends of the tube. The pressure at the beginning P_1 causes force F_1 over area A_1. This force is applied over distance $v_1 dt$ in time dt, so the input energy is

$$dE_1' = P_1 A_1 v_1 dt$$

Similarly for the energy emergent from the tube

$$dE_2' = P_2 A_2 v_2 dt$$

By conservation of energy

$$dE_2 - dE_1 = dE_1' - dE_2'$$

$$\rho_1 v_1 A_1 dt \left(\frac{1}{2}v_2^2 + u_2 + gh_2 - \left(\frac{1}{2}v_2^1 + u_1 + gh_1\right)\right) = \left(P_1 A_1 v_1 - P_2 A_2 v_2\right) dt$$

$$\rho_1 v_1 A_1 \left(\frac{1}{2}v_2^2 + u_2 + gh_2 - \left(\frac{1}{2}v_2^1 + u_1 + gh_1\right)\right) = \left(\frac{P_1}{\rho_1}\rho_1 A_1 v_1 - \frac{P_2}{\rho_2}\rho_2 A_2 v_2\right)$$

By conservation of mass, $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$, as already mentioned. So

$$\frac{1}{2}v_2^2 + u_2 + gh_2 - \left(\frac{1}{2}v_1^2 - u_1 + gh_1\right) = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}$$

Hence, we find that along the streamtubes

$$\frac{P}{\rho} + \frac{1}{2}v^2 + u + gh = const. \tag{7}$$

This is the most general Bernoulli equation. For incompressible fluids, we can further impose condition that ρ and u are constant, and therefore rewrite the Bernoulli equation as

$$P + \frac{1}{2}\rho v^2 + \rho gh = const. \tag{8}$$

Now, a few examples of use of Bernoulli equation are discussed

2.1 Archimedes Law

Suppose we have a stationary fluid so that $v = 0$ everywhere. We want to find the force created by pressure P on some volume V. The force acting on the small vector surface area $d\vec{S}$ pointing outwars is $d\vec{F} = -Pd\vec{S}$. Hence the total force is

$$\vec{F} = -\oiint_S P d\vec{S}$$

By Bernoulli equation

$$P + \rho gh = C$$

where C is some constant
Hence

$$\vec{F} = \oiint_S (\rho gh - C) d\vec{S}$$

Using Gauss's theorem for scalars for volume V closed by surface S

$$\vec{F} = \iiint_V \nabla(\rho gh - C) dV = \hat{h} \iiint_V \rho g dV \tag{9}$$

where $\hat{h}$ is the unit vector in the direction of h. For constant ρ and g we then have the familiar form

$$\vec{F} = \hat{h}\rho g V = -\rho V \vec{g} \tag{10}$$

2.2 Hydrostatic Pressure

For a static fluid, the pressure difference between two points can be found using the Bernoulli equation as

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$
$$\Delta P = -\rho g \Delta h \tag{11}$$

2.3 Basic Lift

Consider an aerofoil in incompressible air. The air flows at speed v_t over the top of the aerofoil and at speed v_b over the bottom of the foil. Bernoulli equation taken from point far upstream gives us

$$P_0 + \frac{1}{2}\rho v_0^2 = P_t + \frac{1}{2}\rho v_t^2 = P_b + \frac{1}{2}\rho v_b^2$$

This means that

$$P_b - P_t = \frac{1}{2}\rho(v_t^2 - v_b^2)$$

Hence, if the foil causes a lift, the speed at the top of the foil must be bigger than the speed at the bottom of the foil.

2.4 Shallow Water Waves

Consider steady waves forming in shallow water of depth h with wave amplitude A. If the waves move at speed v in the x direction, they appear stationary if we transform by Galilei transformations into frame moving at v in x direction. However, the water in this frame moves by average speed $-v$ in the x direction. However, consider the mass conservation in area of width w, perpendicular to the direction of average fluid velocity and parallel to average surface.
Let the speed at the top of the wave be v_1 and at the bottom v_2. Mass conservation requires that

$$\rho(A+h)wv_1 = \rho(A-h)wv_2$$

$$(A+h)v_1 = (h-A)v_2$$

$$(v_1+v_2)A = h(v_2-v_1)$$

Or, using the average speed v

$$2vA = h(v_2-v_1)$$

Using the Bernoulli equation between these two points

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g(h+A) = P_2 + \frac{1}{2}\rho v_2^2 + \rho g(h-A)$$

Since the surface has to be at atmospheric pressure P_0, we have $P_1 = P_2 = P_0$. Therefore

$$\frac{1}{2}\rho(v_2^2 - v_1^2) = \rho g(h+A-(h-A))$$

$$\frac{v_1+v_2}{2}(v_2-v_1) = 2gA$$

Using the equation for mass conservation

$$v2vA = 2gAh$$

$$v^2 = gh$$

$$v = \sqrt{gh}$$

This is a classical result and means that waves get faster as they approach the shore. Also, it means that the top of the tall waves travels faster than the bottom, which leads to breaking waves.

2.5 Ideal Gas

For ideal gas, the internal energy mass density is

$$u = \frac{U}{m} = \frac{C_V T}{M}$$

where C_V is the molar heat capacity and M is the molar mass and T is the temperature. Also, for gas

$$\frac{P}{\rho} = \frac{PV}{m} = \frac{nRT}{m} = \frac{RT}{M} = \frac{(C_p - C_V)T}{M}$$

Hence, the general Bernoulli equation becomes

$$\frac{(C_p - C_V)T}{M} + \frac{1}{2}v^2 + \frac{C_V T}{M} + gh = const.$$

and

$$\frac{C_p T}{M} + \frac{1}{2}v^2 + gh = const. \tag{12}$$

This means that at the same potential height, the colder fluid flows faster.

3 Fluid Vector Analysis

For now, we mainly applied scalar calculus to the study of fluids. Now, we will try to apply some vector calculus to the velocity field of the flow in the fluids. We start by the incompressibility in the flow.

3.1 Incompressibility and Continuity Equation

Consider velocity field $\vec{v}$. Let V be some volume enclosed by surface S, somewhere in the flow.
The mass entering the volume V per unit time is given by

$$\oiint_S \rho\vec{v}\cdot(-d\vec{S}) = -\oiint_S \rho\vec{v}\cdot d\vec{S}$$

The mass entering the volume is the mass change per unit time - i.e. the derivative of mass with respect to time. For this derivative

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Equating these two

$$-\oiint_S \rho\vec{v}\cdot d\vec{S} = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Using Gauss's law

$$\iiint_V \nabla\cdot(-\rho\vec{v})dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Since the integrals are over the same volume, the integrands must be equal. Therefore, we have the continuity equation

$$\nabla\cdot(\rho\vec{v}) = -\frac{\partial \rho}{\partial t} \tag{13}$$

But, for incompressible fluids, ρ is constant and therefore

$$\nabla\cdot\vec{v} = 0 \tag{14}$$

3.2 Fluid Acceleration

Consider a velocity field $\vec{v}(\vec{r},t)$ where $\vec{r}$ is the position vector. The velocity field effectively prescribes the velocity to the particles at every point in space and time. So, what is an acceleration of a particle so that it follows this velocity field? Consider particle at point $(\vec{r},t)$. In small time dt, particle moves to the point $(\vec{r}+\vec{v}(\vec{r},t)dt, t+dt)$. Here, the speed of the particle is $\vec{v}' = \vec{v}(\vec{r}+\vec{v}dt, t+dt)$. Hence, the change in speed of the particle is

$$\frac{D\vec{v}}{Dt} = \lim_{dt\to 0}\frac{\vec{v}'-\vec{v}}{dt} = \lim_{dt\to 0}\frac{\vec{v}(\vec{r}+\vec{v}dt,t+dt)-\vec{v}(\vec{r},t)}{dt} =$$

$$\lim_{dt\to 0}\frac{1}{dt}\left(\frac{\partial\vec{v}}{\partial x}v_x dt + \frac{\partial\vec{v}}{\partial y}v_y dt + \frac{\partial\vec{v}}{\partial z}v_z dt + \frac{\partial\vec{v}}{\partial t}dt\right) = (\vec{v}\cdot\nabla + \frac{\partial}{\partial t})\vec{v}$$

Therefore, we can represent the operator $\frac{D}{Dt}$ as

$$\frac{D}{Dt} = \vec{v}\cdot\nabla + \frac{\partial}{\partial t} \tag{15}$$

This operator is called the conductive derivative and represents the acceleration of particles at given point of the velocity field.

3.3 Navier-Stokes' Equations

We can now use a Newton's second law to figure out the differential equations of fluid motion.
First, we have to consider all forces acting on a small element of volume inside the fluid, dV.
First type of forces are the forces due to pressure.

3.3.1 Pressure Forces

Imagine dV is a cube with faces parallel to cartesian unit planes. The force in the x direction on the fluid inside the volume is

$$F_x = P(x,y,z)dydz - P(x+dx,y,z)dydz$$

In the limit of a small cube, this becomes

$$F_x = -\frac{\partial P}{\partial x}dxdydz = -\frac{\partial P}{\partial x}dV$$

Similarly, we could derive

$$F_y = -\frac{\partial P}{\partial y} dV$$

and

$$F_z = -\frac{\partial P}{\partial z} dV$$

Hence, we conclude that the forces due to pressure follow is

$$\vec{F}_p = -\nabla P dV$$

which is also a coordinate free expression.

3.3.2 Gravitational Forces

The gravity attracts the fluid in common way

$$\vec{F}_g = dm\vec{g}$$

where $\vec{g}$ is the acceleration due to gravity. We can rewrite $dm = \rho dV$ to get

$$\vec{F}_g = \rho\vec{g}dV$$

3.3.3 Viscous Forces

Again, the overall force in the x direction due to overall strain in the y direction in small cube is

$$F_{xy} = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y+dy} dxdz - \mu \left.\frac{\partial v_x}{\partial y}\right|_{y} dxdz = \mu \frac{\partial^2 v_x}{\partial y^2} dxdydz = \mu \frac{\partial^2 v_x}{\partial y^2} dV$$

Similarly, the force in x direction due to strain in z direction

$$F_{xz} = \mu \left.\frac{\partial v_x}{\partial z}\right|_{z+dz} dxdy - \mu \left.\frac{\partial v_x}{\partial z}\right|_{z} dxdy = \mu \frac{\partial^2 v_x}{\partial z^2} dV$$

And similarly

$$F_{xx} = \mu \frac{\partial^2 v_x}{\partial x^2} dV$$

Also, for all other components of the force, we would get exactly analogous expressions. Therefore, we have

$$F_i = F_{ix} + F_{iy} + F_{iz} = \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_i dV = \mu \nabla^2 v_i dV$$

Hence, we have

$$\vec{F}_v = \mu \nabla^2 \vec{v} dV$$

3.3.4 Overall Force

Therefore, the overall force on the element of volume dV is

$$\vec{F} = \vec{F}_v + \vec{F}_p + \vec{F}_g = \left(\mu \nabla^2 \vec{v} - \nabla P + \rho \vec{g}\right) dV$$

The acceleration of the fluid in the volume element is

$$\vec{a} = \frac{D\vec{v}}{Dt} = (\vec{v} \cdot \nabla + \frac{\partial}{\partial t})\vec{v}$$

The mass of the fluid in the element is

$$dm = \rho dV$$

Hence, the Newton's second law is

$$dm\vec{a} = \vec{F}$$

$$\rho(\vec{v} \cdot \nabla + \frac{\partial}{\partial t})\vec{v} dV = \left(\mu \nabla^2 \vec{v} - \nabla P + \rho \vec{g}\right) dV$$

And therefore

$$\rho(\vec{v} \cdot \nabla + \frac{\partial}{\partial t})\vec{v} = \mu \nabla^2 v - \nabla P + \rho \vec{g} \tag{16}$$

These equations (in fact three separate equations for each component of the fields) are called the Navier-Stokes' equations. They are second order non-linear differential equations, and they do not have any general solution. Therefore, we will need to employ approximations to predict the fluid behaviour.

3.3.5 Boundary Conditions

Since we have our partial differential equations describing the velocity field, we need to have some boundary conditions to find specific solutions.
Usually, the fluid is enclosed in some form of tube or other form of container. Then, the behaviour of the fluid close to the borders of this container is of interest.
Clearly, if the container is sealing the fluid, it cannot go through the container. Therefore, the component of the velocity field normal to the surface of the container should be zero.
From experiment, we also know that the transverse component of the velocity field should be zero as well. This is of course speaking in terms of relative velocities, and therefore if the object moves, the fluid velocity on the surface must be equal to the velocity of the object.
If the object is stationary, the velocity of the fluid on the surface of the boundary must be zero.

3.3.6 Approximations

Usually, we choose two approximations - high viscosity and high inertia approximation.
We can see that the term $\rho\left(\vec{v}\cdot\nabla+\frac{\partial}{\partial t}\right)\vec{v}$ corresponds to the innertial term (due to the acceleration and speed of the fluid) and $\mu\nabla^2\vec{v}$ presents the viscosity term. The approximate magnitudes of these terms are (for steady flow)

$$\rho\left(\vec{v}\cdot\nabla+\frac{\partial}{\partial t}\right)\vec{v}\approx\rho\frac{v_{av}}{d}v_{av}$$

where v_{av} is the average speed, ρ is the average density and d is the average diameter of the container. Similarly, the viscosity term magnitude is

$$\mu\nabla^2\vec{v}\approx\mu\frac{v_{av}}{d^2}$$

Hence, their ratio is

$$\frac{\rho(\vec{v}\cdot\nabla)\vec{v}}{\mu\nabla^2\vec{v}}\approx\frac{\rho v_{av}d}{\mu}=Re$$

Hence, we see the significance of Reynolds number - it compares the inertial and viscosity terms in the Navier-Stokes' equations. This result is consistent with the prediction that for small Reynolds number (small inertial term, big viscosity term), the flow is laminar, while for high Reynolds number (big inertial term), the flow is turbulent and non-laminar.

3.4 General Poiseuille Flow

From the high viscosity mode of Navier-Stokes' equations, we can derive the more general formula for Poiseuille flow. Assuming that the inertial term is approximately zero, the equations become

$$0\approx-\nabla P+\rho\vec{g}+\mu\nabla^2\vec{v}$$

Hence we have

$$\mu\nabla^2\vec{v}=\nabla P-\rho\vec{g} \tag{17}$$

Or, if the fluid is on the same potential level, or the gravitational energy is negligable

$$\mu\nabla^2\vec{v}=\nabla P \tag{18}$$

We can see that this is exactly analogous to our previously derived expressions for Poiseuille flow.
One important feature of this flow is that it is linear, and therefore much easier to solve than the previous equations.

3.5 Euler Flow

Euler approximation is the opposite approximation of the Poiseuille approximation - it assumes that the viscous term is negligable. Therefore, Navier-Stokes' equations become

$$\rho(\vec{v}\cdot\nabla+\frac{\partial}{\partial t})\vec{v}=-\nabla P+\rho\vec{g} \tag{19}$$

These equations are called the Euler equations. They are first order differential equations, but they are still non-linear.

Importantly, since they are only first order equations, they can only accomodate one boundary condition. But, for experimental consistency, we need to have two boundary conditions on the borders of the containers. But, since at the borders, the viscous effects should cause the fluid to stop, we should perhaps not be too surprised that the Euler model fails this close to the boundary.
Generally, Euler flow models well the behaviour of the fluid when sufficiently far away from any boundary, and hence only has to satisfy the condition that no water flows into the boundary layer.

3.6 Boundary Layer

We stated that Euler approximation is good outside the boundary layer. But, what is the approximate size of this boundary layer? Consider a thin foil of length d that is set in a flow with its long side parallel to the flow. At the boundary layer edge, the inertial term must be close to the viscous term, i.e.

$$\mu\nabla^2\vec{v} \approx \rho(\vec{v}\cdot\nabla)\vec{v}$$

The inertial term corresponds to Euler approximation, hence the change of the velocity here is considered along the length of the thin foil, which is d. On the other hand, the viscous term corresponds starts to change drastically when close to boundary, hence, it corresponds to change over some distance δ, which corresponds to the thickness of the boundary layer. Hence, taking the approximate average values

$$\begin{aligned}
\mu\frac{v_{av}}{\delta^2} &\approx \rho\frac{v_{av}^2}{d} \\
\delta^2 &\approx \frac{\mu d}{\rho v_{av}} \\
\frac{\delta^2}{d^2} &\approx \frac{\mu}{\rho v_{av} d} \\
\frac{\delta}{d} &\approx \sqrt{\frac{1}{Re}} \qquad (20)
\end{aligned}$$

Hence, the relative thickness of the boundary layer increases as the Reynolds number decreases, as expected.

4 Synthetic Flows

Even with the approximations used in the Navier-Stokes' equations themselves, we still need some other approximations to take place. Usually, we choose the incompressibility condition

$$\nabla\cdot\vec{v} = 0$$

Furthemore, we often try to solve cases where the flow is also irrotational, i.e.

$$\nabla\times\vec{v} = 0$$

Usually, this applies for non-turbulent but also not very viscous flow.

4.1 Circulation and Vorticity

Vorticity and circulation are two variables to be consider when defining the curl of the fluid velocity field. The vorticity is defined as

$$\vec{w} = \frac{1}{2}\nabla\times\vec{v}$$

and somehow connects to the rotation of the velocity vector around every point, and the $\frac{1}{2}$ factor basically takes average of this rotational behaviour.
The circulation is defined as

$$K = \oint_\Gamma \vec{v}\cdot d\vec{l}$$

where $d\vec{l}$ is a vector line element along the closed curve Γ. If the surface enclosed by this curve is S, we can use Stokes' theorem to get

$$K = \oint_\Gamma \vec{v}\cdot d\vec{l} = \iint_S (\nabla\times\vec{v})\cdot d\vec{S}$$

where $d\vec{S}$ is the vector element of the area, pointing in a direction given by the right hand rule with respect to orientation of Γ.

4.2 Kelvin's Circulation Theorem

Consider a circulation around a certain loop Γ that moves with the flow. The change in the circulation around the loop is

$$\frac{DK}{Dt} = \frac{D}{Dt}\oint_\Gamma \vec{v}\cdot d\vec{l}$$

Using the product rule (and moving the derivative inside the integral)

$$\frac{DK}{Dt} = \oint_\Gamma \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \oint_\Gamma \vec{v}\cdot\frac{D(d\vec{l})}{Dt}$$

The change of the line element in time is equal to the change in the velocity, as the loop is carried by the stream. Hence

$$\frac{D(d\vec{l})}{Dt} = d\vec{v}$$

Thus, the second integral becomes

$$\oint_\Gamma \vec{v}\cdot\frac{D(d\vec{l})}{Dt} = \oint_\Gamma \vec{v}\cdot d\vec{v} = \frac{1}{2}\oint_\Gamma d(|\vec{v}|^2)$$

Since this is the integral of a differential of a scalar function around a closed loop, this integral equals zero. It is worth noting that the same result would also apply for stationary loop, when $\frac{D(d\vec{l})}{Dt} = \vec{0}$.
Thus, the change in circulation becomes

$$\frac{DK}{Dt} = \oint_\Gamma \left(\vec{v}\cdot\nabla + \frac{\partial}{\partial t}\right)\vec{v}\cdot d\vec{l}$$

Using Euler approximation

$$\frac{DK}{Dt} = \oint_\Gamma \left(-\frac{\nabla P}{\rho} + \vec{g}\right)\cdot d\vec{l}$$

For constant $\vec{g}$ and ρ, this can be rewritten as

$$\frac{DK}{Dt} = \oint_\Gamma -\nabla\left(\frac{P}{\rho} - \vec{g}\cdot\vec{r}\right)\cdot d\vec{l}$$

This is the integral of a gradient of a scalar function over a closed loop, which is always zero. Therefore, we arrive at Kelvin's circulation theorem

$$\frac{DK}{Dt} = 0 \tag{21}$$

This means that the circulation is conserved in the incompressible flow.
Therefore, if we are able to produce a curl free flow, it tends to remain curl free, which makes our approximation at least stability-wise valid.

4.3 Potential Flow

Again, consider an irrotational, incompressible flow. Because it is irrotational, i.e.

$$\nabla\times\vec{v} = 0$$

there exists a scalar function Φ such that

$$\vec{v} = \nabla\Phi$$

Because the flow is incompressible, it means that

$$\nabla\cdot\vec{v} = 0$$

Hence, substituting for the scalar function

$$\nabla\cdot(\nabla\Phi) = \nabla^2\Phi = 0$$

This is the Laplace's equation, which is a linear equation, and we will use the linearity of incompressible, irrotational flow heavily.
In analogy with electrostatics, the scalar function Φ is called the velocity potential. Therefore, the incompressible, irrotational flow is often reffered to as the potential flow.

4.4 Streamfunction

Simply from the incompressibility condition $\nabla \cdot \vec{v} = 0$, it follows that

$$\vec{v} = \nabla \times \vec{\Psi}$$

where $\vec{\Psi}$ is the so called streamfunction. Interesingly, for irrotational flow, it follows that

$$\nabla \times \vec{v} = 0$$

$$\nabla \times (\nabla \times \vec{\Psi}) = \nabla(\nabla \cdot \vec{\Psi}) - \nabla^2 \vec{\Psi} = 0$$

Again, in analogy with electromagnetism, by choosing a Coulomb gauge $\nabla \cdot \Psi = 0$, we have again the Laplace's equation, but this time in vector form

$$\nabla^2 \vec{\Psi} = 0$$

Specifically, in 2D case, we can assign Ψ to have only one component - in the z direction, as this is the only one that matters for the velocity arguments. Hence, in 2D, the streamfunction becomes effectively a scalar, and can be used to describe potential flows in very similar way as the velocity potential.

4.5 Examples of Potentail Flow

4.5.1 Uniform flow in 2D

Consider a uniform flow $\vec{v} = v\hat{i}$ in 2D. The velocity potential producing this movement follows

$$\nabla \Phi = v\hat{i}$$

Hence

$$\Phi = vx + C$$

where C is some constant.
The streamfunction follows (since only z component is important)

$$v = \frac{\partial \Psi_z}{\partial y}$$

$$\Psi_z = vy + C'$$

where C' is some other constant.

4.5.2 Point Source/Sink

Consider a point source of fluid centered at the origin. Due to symmetry, the integral over the circle S centered at the origin of the velocity $\vec{v}$ is (in 2D)

$$\oiint_S \vec{v} \cdot d\vec{S} = 2\pi R v$$

where $\vec{v}(R) = v\hat{e}_r$ and R is the radius of the circular surface and $d\vec{S}$ points perpendicularly outwards of the circle.
The point source will have effect as in electromagnetism, i.e.

$$\iiint_V \nabla \cdot \vec{v} dV = Q$$

where Q is the flow of the liquid into the system (in 2D in meters squared per second). Thus, by Gauss's law

$$Q = 2\pi R v$$

And therefore

$$\vec{v} = \frac{Q}{2\pi r}\hat{e}_r \tag{22}$$

The potential follows

$$\nabla \Phi = \frac{Q}{2\pi r}\hat{e}_r = \frac{Q}{2\pi}\nabla(\ln(r))$$

Therefore

$$\Phi = \frac{Q}{2\pi}\ln(r) + C$$

where C is some constant.
The streamfunction is

$$\nabla \times \Psi = \frac{Q}{2\pi r}\hat{e}_r$$

Since only the z component is present, we can use the form of curl in cylindrical coordinates

$$\frac{Q}{2\pi r} = \frac{1}{r}\frac{\partial \Psi_z}{\partial \phi}$$

$$\Psi_z = \frac{Q}{2\pi}\phi + C'$$

where C' is some constant.

4.6 Potential Bernoulli Equation

In order to understand the derivation of Bernoulli equation in terms of the velocity potential, we first need to derive a vector calculus formula for the gradient of a scalar product of two vectors.

4.6.1 Gradient of Scalar Product

Lets have two vectors $\vec{a}$ and $\vec{b}$, with cartesian components $a_1 = a_x, a_2 = a_y, a_3 = a_z$ and similarly for $\vec{b}$. Also, lets define that $a_n = a_{n \mod 3}$, so that for example $a_5 = a_2$ and $a_{-1} = a_2$. Similarly, let $x_1 = x, x_2 = y, x_3 = z$ and $x_n = x_{n \mod 3}$.
Now, lets consider ith component of vector $\vec{a} \times (\nabla \times \vec{b})$

$$(\vec{a} \times (\nabla \times \vec{b}))_i = a_{i+1}(\nabla \times b)_{i+2} - a_{i+2}(\nabla \times b)_{i+1} = a_{i+1}\left(\frac{\partial b_{i+4}}{\partial x_{i+3}} - \frac{\partial b_{i+3}}{\partial x_{i+4}}\right) - a_{i+2}\left(\frac{\partial b_{i+3}}{\partial x_{i+2}} - \frac{\partial b_{i+2}}{\partial x_{i+3}}\right)$$

Applying the modulo identities

$$(\vec{a} \times (\nabla \times \vec{b}))_i = a_{i+1}\left(\frac{\partial b_{i+1}}{\partial x_i} - \frac{\partial b_i}{\partial x_{i+1}}\right) - a_{i+2}\left(\frac{\partial b_i}{\partial x_{i+2}} - \frac{\partial b_{i+2}}{\partial x_i}\right) =$$

$$= a_{i+1}\frac{\partial}{\partial x_i}b_{i+1} + a_{i+2}\frac{\partial}{\partial x_i}b_{i+2} - \left(a_{i+1}\frac{\partial}{\partial x_{i+1}} + a_{i+2}\frac{\partial}{\partial x_{i+2}}\right)b_i$$

Adding $a_i\frac{\partial}{\partial x_i}b_i - a_i\frac{\partial}{\partial x_i}b_i = 0$ to the right side of the equation, we are left with

$$(\vec{a} \times (\nabla \times \vec{b}))_i = a_i\frac{\partial}{\partial x_i}b_i + a_{i+1}\frac{\partial}{\partial x_{i+1}}b_{i+1} + a_{i+2}\frac{\partial}{\partial x_{i+2}}b_{i+2} - \left(a_i\frac{\partial}{\partial x_i} + a_{i+1}\frac{\partial}{\partial x_{i+1}} + a_{i+2}\frac{\partial}{\partial x_{i+2}}\right)b_i =$$

$$= a_i\frac{\partial}{\partial x_i}b_i + a_{i+1}\frac{\partial}{\partial x_{i+1}}b_{i+1} + a_{i+2}\frac{\partial}{\partial x_{i+2}}b_{i+2} - (\vec{a} \cdot \nabla)b_i$$

We can proceed exactly analogously but reverse the role of $\vec{a}$ and $\vec{b}$ to get

$$(\vec{b} \times (\nabla \times \vec{a}))_i = b_i\frac{\partial}{\partial x_i}a_i + b_{i+1}\frac{\partial}{\partial x_{i+1}}a_{i+1} + b_{i+2}\frac{\partial}{\partial x_{i+2}}a_{i+2} - (\vec{b} \cdot \nabla)a_i$$

We can see that the first three terms in both these expressions together create the gradient of scalar product

$$\nabla(\vec{a} \cdot \vec{b})_i = \frac{\partial}{\partial x_i}(a_i b_i + a_{i+1}b_{i+1} + a_{i+2}b_{i+2}) =$$

$$= a_i\frac{\partial}{\partial x_i}b_i + a_{i+1}\frac{\partial}{\partial x_{i+1}}b_{i+1} + a_{i+2}\frac{\partial}{\partial x_{i+2}}b_{i+2} + b_i\frac{\partial}{\partial x_i}a_i + b_{i+1}\frac{\partial}{\partial x_{i+1}}a_{i+1} + b_{i+2}\frac{\partial}{\partial x_{i+2}}a_{i+2}$$

Hence, we have

$$(\vec{a} \times (\nabla \times \vec{b}))_i + (\vec{b} \times (\nabla \times \vec{a})) = \nabla(\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \nabla)b_i - (\vec{b} \cdot \nabla)a_i$$

Since this applies for all i, we have a vector identity

$$\nabla(\vec{a} \cdot \vec{b}) = \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{b}) + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a} \tag{23}$$

And, in the special case of $\vec{a} = \vec{b}$, this becomes

$$\nabla(|\vec{a}|^2) = 2\vec{a} \times (\nabla \times \vec{a}) + 2(\vec{a} \cdot \nabla)\vec{a} \tag{24}$$

4.6.2 Velocity Potential in Euler Equations

Euler equations can be written as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\left(\frac{P}{\rho} - \vec{g}\cdot\vec{r}\right)$$

if $\vec{g}$ is a constant. For irrotational flow

$$(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\nabla(v^2) - \vec{v}\times(\nabla\times\vec{v}) = \frac{1}{2}\nabla(v^2)$$

Therefore, we have

$$\frac{\partial \vec{v}}{\partial t} = -\nabla\left(\frac{P}{\rho} + \frac{1}{2}v^2 - \vec{g}\cdot\vec{r}\right)$$

For potential flow, $\vec{v} = \nabla\Phi$, and therefore

$$-\nabla\left(\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}v^2 - \vec{g}\cdot\vec{r}\right) = \vec{0}$$

This means that

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}v^2 - \vec{g}\cdot\vec{r} = const. \tag{25}$$

Which is the formulation of bernoulli equation for irrotational potential flow in constant gravitational field.

4.7 Potential Vortex

Having no curl in the $\vec{v}$ field is useful from a mathematical standpoint, but causes problems when we need to model real fluids. Fortunately, there exists a trick how to model even circulating systems without creating a curl in the velocity field. This trick is introducing a potential vortex into the field.
Assume that we have some form of circulation with cylindrical symmetry, so that $\vec{v} = v_r(r)\hat{e}_r + v_\phi(r)\hat{e}_\phi + v_z(r)\hat{e}_z$ and $v_r = v_z = 0$ i.e. the fluid only circulates around the z axis. We require that the curl of this field is zero at every point where it is defined. Therefore

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial\phi}\right)\hat{e}_z = 0$$

Applying our conditions, we are left with

$$\nabla\times\vec{v} = \frac{1}{r}\frac{\partial(rv_\phi)}{\partial r} = 0$$
$$rv_\phi = C$$
$$v_\phi = \frac{C}{r}$$

Therefore, the field $\vec{v} = v_\phi\hat{e}_\phi = \frac{C}{r}\hat{e}_\phi$ has zero curl even though it somehow rotates. The critical question is now whether the circulation of the field is also zero. Consider circulation around a circular loop in the xy plane of radius r. The circulation is

$$K = \oint_\Gamma \vec{v}\cdot d\vec{l} = \int_0^{2\pi}\vec{v}\cdot(rd\phi\hat{e}_\phi) = \int_0^{2\pi} v_\phi r d\phi = v_\phi 2\pi r$$

Hence

$$v_\phi = \frac{K}{2\pi r}$$

Which is in agreement with our previous result if $C = \frac{K}{2\pi}$. But, this means that K is non-zero. How is this possible if we could write, by Stokes' theorem, the previous integral as an integral over the surface of the curl of the velocity? It is simply possible because the $\vec{v}$ field, and therefore its curl, is not defined along the z axis ($r = 0$, so the field diverges). This enables for this otherwise non-standard behaviour.

4.7.1 Pressure and Height Drop In Vortex

Consider now a real stable vortex in the fluid. Let the flow be incompressible and irrotational. We expect that the very core of the vortex will behave differently, as v_ϕ will increase a lot and thus the viscous effects will probably take place at some point (because μ will stop being a constant for these velocities).
Outside the core, the Bernoulli equation and potential vortex rules apply. Consider now a streamtube from very distant point at the surface of the fluid to the point on the surface of the fluid close to the core, but still described by potential vortex. The Bernoulli equation states that (for stationary flow)

$$\frac{P_0}{\rho} = \frac{P(r)}{\rho} + \frac{1}{2}v(r)^2$$

where P_0 is the pressure in the fluid on the surface far away from the core, ρ is the density, $P(r)$ is the pressure at distance r from the core and $v(r) = v_\phi(r)$ is the magnitude of the velocity at distance r away from the core. Using the potential vortex rules as derived above

$$P(R) = P_0 - \frac{\rho}{2}\frac{K^2}{4\pi^2 r^2} = P_0 - \frac{K^2\rho}{8\pi^2 r^2}$$

This means that the pressure near the core starts rapidly dropping. But, in real world, the atmosphere keeps the pressure at the surface of the fluid at constant value of P_0. This means that the surface at the vortex is pushed down by the atmosphere. Therefore, we need to alter our Bernoulli equation to

$$\frac{P_0}{\rho} + gH = \frac{P_0}{\rho} + \frac{1}{2}v(r)^2 + g(H - h(r))$$

where H is the depth of the water and $h(r)$ is the depth of the vortex at given point. Therefore

$$gh(r) = \frac{K^2}{8\pi^2 r^2}$$

$$h(r) = \frac{K^2}{8\pi^2 g r^2}$$

4.7.2 Potential and Streamfunction of Potential Vortex

Potential vortex is $\vec{v} = \frac{K}{2\pi r}\hat{e}_\phi = \nabla\Phi$. In cylindrical coordinates

$$\frac{1}{r}\frac{\partial\Phi}{\partial\phi} = \frac{K}{2\pi r}$$

$$\Phi = \frac{K}{2\pi}\phi + C$$

where C is some constant.
For the streamfunction, this flow is essentially 2D. Therefore, we can only consider the z component of the streamfunction. For curl in cylindrical coordinates

$$-\frac{\partial\Psi_z}{\partial r} = \frac{K}{2\pi r}$$

$$\Psi_z = -\frac{K}{2\pi}\ln(r) + C'$$

where C' is some constant. Notice the similarity to the source/sink relationships for potentail and streamfunction - since the potential vortex is purely rotational motion and source is purely divergent motion, the potential and streamfunction effectively exchanged their forms, up to a -1 factor in front of the streamfunction now.

4.8 Principle of Superposition

We already stated that if the flow is irrotational and incompressible, then the potentail and streamfunction both exist and follow Laplace's equations, which are linear and therefore we can use the principle of superposition to find more complicated flows from the cases of easy flows. Is this also reflected in form of Navier-Stokes' equations for incompressible and irrotational flow?

The equations are

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{\nabla P}{\rho}+\frac{\mu}{\rho}\nabla^2\vec{v}+\vec{g}$$

But

$$(\vec{v}\cdot\nabla)\vec{v}=\frac{1}{2}\nabla(v^2)-\vec{v}\times(\nabla\times\vec{v})=\frac{1}{2}\nabla(v^2)$$

since the flow is irrotational.
Also

$$\nabla^2 v=\nabla(\nabla\cdot\vec{v})-\nabla\times(\nabla\times\vec{v})=0$$

since the flow is both irrotational and incompressible. Therefore, the Navier-Stokes' equations are

$$\frac{\partial\vec{v}}{\partial t}+\frac{1}{2}\nabla(v^2)=-\frac{\nabla P}{\rho}+\vec{g}$$

And thus we can see that if the flow is stationary ($\frac{\partial\vec{v}}{\partial t}=0$), the equations are linear in v^2, and thus the principle of superposition is applicable.

4.8.1 Velocity Field Around a Cylinder in a Uniform Flow in 2D

Suppose we have a cylinder (effectively a circle, since we work in 2D) centered at the origin with radius R. The uniform flow has velocity v and the x direction, so the uniform flow is $\vec{v}_u=v\hat{i}$.
We try to model the behaviour of the flow around the boundary by ensuring that at the boundary, the velocity of the flow is zero. This can be achieved with source/sink pair inside the cylinder, symmetrically placed around the origin on the x axis at positions $(dx,0)$ for the source and $(-dx,0)$ for the sink (using cartesian coordinates). The streamfunction from a point source at the origin is

$$\Psi=\frac{Q}{2\pi}\phi=\frac{Q}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

For small displacement da in the x direction, this becomes

$$\Psi'=\frac{Q}{2\pi}\tan^{-1}\left(\frac{y}{x+da}\right)=\frac{Q}{2\pi}\tan^{-1}\left(\frac{y}{x}\frac{1}{1+\frac{da}{x}}\right)\approx\frac{Q}{2\pi}\tan^{-1}\left(\frac{y}{x}\left(1-\frac{da}{x}\right)\right)=\frac{Q}{2\pi}\tan^{-1}\left(\frac{y}{x}-\frac{y}{x^2}da\right)$$

Using Taylor expansion

$$\Psi'\approx\frac{Q}{2\pi}\left[\tan^{-1}\left(\frac{y}{x}\right)+\left.\frac{\partial(\tan^{-1}(z))}{\partial z}\right|_{z=\frac{y}{x}}\left(-\frac{y}{x^2}da\right)\right]=\frac{Q}{2\pi}\left[\phi-\frac{1}{1+\frac{y^2}{x^2}}\frac{y}{x^2}da\right]=\frac{Q}{2\pi}\left(\phi-\frac{y}{x^2+y^2}da\right)$$

Substituting for y and x

$$\Psi'=\frac{Q}{2\pi}\left(\phi-\frac{r\sin\phi}{r^2}da\right)=\frac{Q}{2\pi}\left(\phi-\frac{\sin\phi da}{r}\right)$$

So, the streamfunction due to the source is

$$\Psi_+=\frac{Q}{2\pi}\left(\phi-\frac{\sin\phi dx}{r}\right)$$

And due to the sink (placed at $da=-dx$ and has negative flow $Q=-Q$)

$$\Psi_-=\frac{-Q}{2\pi}\left(\phi+\frac{\sin\phi dx}{r}\right)$$

The streamfunction that produces the uniform flow $\vec{v}=v\hat{i}$ is (reffering just to the z component) such that

$$\frac{\partial\Psi_u}{\partial y}=v$$

$$\Psi_u=vy$$

Thus, the overall streamfunction is

$$\Psi=\Psi_u+\Psi_++\Psi_-=vr\sin\phi-\frac{Q}{\pi r}\sin\phi dx$$

The velocity is therefore

$$\vec{v} = \nabla \times \Psi = \frac{1}{r}\frac{\partial \Psi}{\partial \phi}\hat{e}_r - \frac{\partial \Psi}{\partial r}\hat{e}_\phi$$

$$\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = \frac{1}{r}\left(vr\cos\phi - \frac{Q}{\pi r}\cos\phi dx\right) = \cos\phi\left(v - \frac{Q}{\pi r^2}dx\right)$$

Similarly

$$-\frac{\partial \Psi}{\partial r} = -v\sin\phi - \frac{Q}{\pi r^2}\sin\phi dx = -\sin\phi\left(v + \frac{Q}{\pi r^2}dx\right)$$

Therefore

$$\vec{v} = \cos\phi\left(v - \frac{Q}{\pi r^2}dx\right)\hat{e}_r - \sin\phi\left(v + \frac{Q}{\pi r^2}\right)\hat{e}_\phi$$

We therefore see that we cannot satisfy both boundary conditions by just this setup. We can choose to try to satisfy at least impermeability of the cylinder, so that

$$v_r(R) = 0$$

This reqires

$$\cos\phi\left(v - \frac{Q}{\pi R^2}dx\right) = 0$$

$$v = \frac{Q}{\pi R^2}dx$$

Or

$$Qdx = \pi v R^2$$

Thus, we get rid of virtual sink and source parameters in the solution, which is now

$$\vec{v} = \cos\phi\left(v - \frac{\pi v R^2}{\pi r^2}\right)\hat{e}_r - \sin\phi\left(v + \frac{\pi v R^2}{\pi r^2}\right)\hat{e}_\phi$$

Or

$$\vec{v} = \cos\phi v\left(1 - \frac{R^2}{r^2}\right)\hat{e}_r - \sin\phi v\left(1 + \frac{R^2}{r^2}\right)\hat{e}_\phi \tag{26}$$

We can see that at limit of big r, this reduces to

$$\vec{v}(r >> R) \approx \cos\phi v\hat{e}_r - \sin\phi v\hat{e}_\phi = v\hat{i}$$

which is the original flow (thus the situation is perturbative, which is always good to handle). The problem remains with the breaking of the second boundary condition.
We can also calculate the pressure at the boundary from the Bernoulli equation from some point far upstream to the point on the boundary as (neglecting gravity - assuming whole fluid in the same level)

$$P_0 + \frac{1}{2}\rho v^2 = P(R,\phi) + \frac{1}{2}\rho(v_r(R,\phi)^2 + v_\phi(R,\phi)^2)$$

$$P(R,\phi) = P_0 + \frac{1}{2}\rho v^2\left(1 - \cos^2\phi\left(1 - \frac{R^2}{R^2}\right)^2 - \sin^2\phi\left(1 + \frac{R^2}{R^2}\right)^2\right) =$$

$$= P_0 + \frac{1}{2}\rho v^2\left(1 - 4\sin^2\phi\right)$$

Thus, the pressure around the cylinder is symmetrical, which also means that the overall force on the cylinder is zero. This is clearly not what happens, and the reason for this is the turbulence that occurs behind the cylinder (behind in the sense of velocity $v\hat{i}$). This turbulence creates an area of lower pressure behind the cylinder, which then leads to overall force pushing the cylinder in the direction of the flow.

5 Magnus Effect and Vortices

We can describe some of the real flows using a potential flow analysis, but we usually need to add a potential vortex to somehow insert the non-zero circulation into the model. We will now discover a few situations where this is applied.

5.1 Kutta-Zhukovsky Theorem

Consider again the cylinder inside the uniform flow in x direction with speed v. We have shown that

$$\vec{v} = \cos\phi v\left(1-\frac{R^2}{r^2}\right)\hat{e}_r - \sin\phi v\left(1+\frac{R^2}{r^2}\right)\hat{e}_\phi$$

Which followed from a streamfunction of form

$$\Psi = vr\sin\phi - v\sin\phi\frac{R^2}{r}$$

We now imagine that the cylinder is spinning in such a way that it creates a potential vortex effect centered on the origin with circulation K (spinning and hence circulating anti-clockwise - in the positive direction of ϕ). As we derived, the streamfunction just from this circulation is

$$\Psi_C = -\frac{K}{2\pi}\ln(r)$$

and thus the total streamfunction is

$$\Psi = v\sin\phi\left(r-\frac{R^2}{r}\right) - \frac{K}{2\pi}\ln(r)$$

The speed in the $\hat{e}_r$ direction remains unchanged as

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\phi} = v\cos\phi\left(1-\frac{R^2}{r^2}\right)$$

The speed in the ϕ direction changes to

$$v_\phi = -\frac{\partial\Psi}{\partial r} = -v\sin\phi\left(1+\frac{R^2}{r^2}\right) + \frac{K}{2\pi r}$$

Hence, the velocity at the surface of the cylinder is

$$\vec{v}(R,\phi) = v_\phi\hat{e}_\phi = \left[-v\sin\phi\left(1+\frac{R^2}{r^2}\right) + \frac{K}{2\pi r}\right]\hat{e}_\phi$$

We can again use Bernoulli equation to find the pressure at the surface of the cylinder as

$$P_0 + \frac{1}{2}\rho v^2 = P(R,\phi) + \frac{1}{2}p|\vec{v}(R,\phi)|^2$$

$$P(R,\phi) = P_0 + \frac{1}{2}\rho\left(v^2 - v_\phi(R,\phi)^2\right) = P_0 + \frac{1}{2}\rho\left(v^2 - \left(-2v\sin\phi + \frac{K}{2\pi R}\right)^2\right) =$$

$$= P_0 + \frac{1}{2}\rho\left(v^2 - 4v^2\sin^2\phi + 4v\sin\phi\frac{K}{2\pi R} - \frac{K^2}{4\pi^2R^2}\right)$$

We can see that the pressure is now not symmetrical around the cylinder, and therefore there is an overall force on the cylinder. Let the cylinder be long L in the z direction. The force acting on the cylinder can be found as

$$\vec{F} = \iint_S P(R,\phi)(-d\vec{S})$$

where $d\vec{S}$ is the element of surface of the cylinder S, pointing outwards. Thus

$$\vec{F} = -\int_0^L\int_0^{2\pi} P(R,\phi)R\hat{e}_r(\phi)d\phi dz$$

The integration in z is straightforward

$$\vec{F} = -LR\int_0^{2\pi} P(R,\phi)\hat{e}_r(\phi)d\phi$$

In notation of cartesian coordinate unit vectors, $\hat{e}_r = \cos\phi\hat{i} + \sin\phi\hat{j}$, and therefore

$$\vec{F} = -LR\hat{i}\int_0^{2\pi} P(R,\phi)\cos\phi d\phi - LR\hat{j}\int_0^{2\pi} P(R,\phi)\sin\phi d\phi = -LR\hat{i}I_x - LR\hat{j}I_y$$

Here

$$I_x = \int_0^{2\pi}\cos\phi\left[P_0 + \frac{1}{2}\rho\left(v^2 - 4v^2\sin^2\phi + 4v\sin\phi\frac{K}{2\pi R} - \frac{K^2}{4\pi^2R^2}\right)\right]$$

Since $\int_0^{2\pi} C\cos\phi d\phi = 0$ where C is a constant

$$I_x = -2\rho v^2\int_0^{2\pi}\cos\phi\sin^2\phi d\phi + 2\rho v\frac{K}{2\pi R}\int_0^{2\pi}\sin\phi\cos\phi d\phi =$$

$$= -2\rho v^2\left[\frac{\sin^3\phi}{3}\right]_0^{2\pi} + 2\rho v\frac{K}{2\pi R}\left[\frac{\sin^2\phi}{2}\right]_0^{2\pi} = 0$$

For the second integral

$$I_y = \int_0^{2\pi}\sin\phi\left[P_0 + \frac{1}{2}\rho\left(v^2 - 4v^2\sin^2\phi + 4v\sin\phi\frac{K}{2\pi R} - \frac{K^2}{4\pi^2R^2}\right)\right]d\phi$$

Again, since $\int_0^{2\pi} C\sin\phi d\phi = 0$ for constant C, we have

$$I_y = -2\rho v^2\int_0^{2\pi}\sin^3\phi d\phi + 2\rho v\frac{K}{2\pi R}\int_0^{2\pi}\sin^2\phi d\phi$$

Substituting $\sin^3\phi = (1-\cos^2\phi)\sin\phi$

$$I_y = -2\rho v^2\int_0^{2\pi}\sin\phi d\phi + 2\rho v^2\int_0^{2\pi}\sin\phi\cos^2\phi d\phi + 2\rho v\frac{K}{2\pi R}\int_0^{2\pi}\frac{1-\cos(2\phi)}{2}d\phi =$$

$$= 2\rho v^2\left[-\frac{\cos^3\phi}{3}\right]_0^{2\pi} + \rho v\frac{K}{2\pi R}\int_0^{2\pi}d\phi - \rho v\frac{K}{2\pi R}\int_0^{2\pi}\cos(2\phi)d\phi = \rho v\frac{K}{R}$$

Hence the force

$$\vec{F} = -LR\hat{j}I_y = -LR\rho v\frac{K}{R}\hat{j} = L\rho(\vec{v}\times\vec{K})$$

where $\vec{K}$ is defined as pointing in positive z direction for positive circulation and with magnitude K, and $\vec{v} = v\hat{i}$ is the uniform flow field.
This force taken in the units per length in the z direction L. Such force is then labeled $\vec{L}$ and follows

$$\vec{L} = \rho(\vec{v}\times\vec{K}) \tag{27}$$

This theorem leads to so called Magnus effect - rotating objects in a uniform flow (or rotating and translating objects) are under the effect of Magnus force, which is nothing else than integrated Kutta-Zhukovsky force density $\vec{L}$.

5.2 Vortex Line

Vortex line is a curve inside the fluid that forms the axis of number of planar potential vortices. If this is curve is a straight line, than the vortex is cylindrical as discussed above, otherwise it is somehow deformed cylindrical vortex. Importantly, since the vortex line is the axis of all planar vortices, the divergent points (where $v\phi\to\infty$) of these vortices lie on the vortex line.
Now, imagine that we want to find the circulation of the vortex line at two different points on the vortex line, A and B. We can form two perpendicular planes to the vortex line at points A and B and do a circular integral in given plane along the circles Γ_A and Γ_B, with the same radius R, starting at points S_A and S_B, respectively. This way, we would obtain

$$K_A = \oint_{\Gamma_A}\vec{v}\cdot d\vec{l}$$

and

$$K_B = \oint_{\Gamma_B}\vec{v}\cdot d\vec{l}$$

Now, imagine instead a different path, going first along Γ_A, then from S_A to S_B, then along Γ'_B (similar circle, but as discussed below, it has opposite direction) and then back to S_A. This way, the surface that is spanned by this path is a tube-like shell extended along the vortex line, ended by circles Γ_A and Γ'_B. Such surface does not have a point where $\vec{v}$ diverges, as this surface does not cross the vortex line. Therefore, this integral must be equal to zero. Writing the integral explicitly

$$0 = \int_{\Gamma_A} \vec{v} \cdot d\vec{l} + \int_{S_A}^{S_B} \vec{v} \cdot d\vec{l} + \int_{\Gamma'_B} \vec{v} \cdot d\vec{l} + \int_{S_B}^{S_A} \vec{v} \cdot d\vec{l}$$

But

$$\int_{S_A}^{S_B} \vec{v} \cdot d\vec{l} + \int_{S_B}^{S_A} \vec{v} \cdot d\vec{l} = 0$$

And therefore

$$\int_{\Gamma_A} \vec{v} \cdot d\vec{l} = -\int_{\Gamma'_B} \vec{v} \cdot d\vec{l}$$

But, if we carefully analyze the second integral, we find a requirement for the path Γ'_B. In order for the surface to be well defined, we need to do the integral in the opposite direction to direction originally used to measure circulation (consider not a path from S_A to S_B and back, but rather leave some small separation. For example, we could choose two close points on a circle Γ_A, S_A and C_A, and analogous points in the same distance on the second circle. Then, the integral along whole the circle is close to integral from S_A to C_A along the circle. From need to complete the path, we see that direction of Γ_B must be opposite to direction of Γ_A). Therefore, we have

$$\int_{\Gamma_A} \vec{v} \cdot d\vec{l} = \int_{\Gamma_B} \vec{v} \cdot d\vec{l}$$

and

$$K_A = K_B$$

Hence, the circulation is conserved along such vortex line.

5.3 Vortex Line Interaction

Now, imagine two vortex lines, one with circulation $\vec{K}_1$, second with circulation $\vec{K}_2$, running parallel to each other and sufficiantly distant from each other so that the velocity field from each of the vortices at the core of the other one can be approximated as a uniform flow. As we mentioned before, close to the vortex core, the speed is so fast that the fluid effectively behaves like a rigid rotating body. Since at one core of the vortex there is a near uniform flow from the other vortex, the cores are under the effect of the Magnus force. Let $\vec{r}_{12}$ be the position vector from one vortex to the other (in one plane). Clearly

$$\vec{r}_{12} = -\vec{r}_{21}$$

The velocity at the core of the second vortex due to the uniform flow cause by the first vortex is

$$\vec{v}_{12} = \frac{\vec{K}_1 \times \hat{r}_{12}}{2\pi|\vec{r}_{12}|} = \frac{\vec{K}_1 \times \vec{r}_{12}}{2\pi r^2}$$

where I substituted $|\vec{r}_{12}| = |\vec{r}_{21}| = r$.
Similarly

$$\vec{v}_{21} = \frac{\vec{K}_2 \times \vec{r}_{21}}{2\pi r^2}$$

Hence the Magnus force acting on the core of the second vortex due to the velocity caused by the first vortex is

$$\vec{F}_{12} = \rho(\vec{v}_{12} \times \vec{K}_2) = \rho\frac{(\vec{K}_1 \times \vec{r}_{12}) \times \vec{K}_2}{2\pi r^2}$$

Similarly, the force on the first vortex due to the second vortex is

$$\vec{F}_{21} = \rho(\vec{v}_{21} \times \vec{K}_1) = \rho\frac{(\vec{K}_2 \times \vec{r}_{21}) \times \vec{K}_1}{2\pi r^2}$$

Since the lines are parallel, we can choose $\vec{K}_1 = K_1\hat{e}_z$ and $\vec{K}_2 = K_2\hat{e}_z$, and let the different directions be only set by the sign of K_1 or K_2. Thus

$$\vec{F}_{12} = \rho\frac{K_1K_2(\hat{e}_z \times \hat{r}_{12}) \times \hat{e}_z}{2\pi r} = \frac{\rho K_1K_2}{2\pi r}(\hat{e}_{\phi 12} \times \hat{e}_z) = \frac{\rho K_1K_2}{2\pi r}\hat{r}_{12}$$

Hence the vortex lines with the same circulation repel, while the ones with the opposite circulation attract.

5.3.1 Stable Vortex Pairs

In order for a vortex pair to become stable, it must move at some drift speed $\vec{v}_d$ that counteracts the speed due to the other vortex, i.e.

$$\vec{v}_{d1} = \vec{v}_{21}$$

and

$$\vec{v}_{d2} = \vec{v}_{12}$$

Then, the relative velocity of the surrounding fluid at the core of each vortex is $\vec{v}_{12/21} - \vec{v}_{d2/d1} = \vec{0}$, and there is no net force on the core.
From the relations above

$$\vec{v}_{d2} = \vec{v}_{12} = \frac{\vec{K}_1 \times \hat{r}_{12}}{2\pi r} = \frac{K_1}{2\pi r}\hat{e}_{\phi 12}$$

and

$$\vec{v}_{d1} = \vec{v}_{21} = \frac{K_2}{2\pi r}\hat{e}_{\phi 21}$$

Since $\hat{e}_{\phi 12} = -\hat{e}_{\phi 21}$, for the same sign of K_1 and K_1 (same direction of circulation for both vortex lines), the velocities are in the opposite directions, which causes the vortices to orbit each other, while for the opposite signs of K_1 and K_2, the two vortices drift in the same direction.

5.3.2 Lift from Circulation

Again, we consider a wing of a plane. Since the air before the plane is assumed to be circulation free, the overall circulation must still be zero when the plane arrives. But, we try to simulate the plane by adding a potential vortex just after the end of the wing, which means that we are adding a circulation to the system. Therefore, we need to require an opposite circulation around the wing, so that the overall circulation remains zero.
This circulation around the wing is also the circulation that is directly observable - the vortex lines forms around the wings, with circulation in the direction of $\hat{v} \times \hat{h}$, where $\hat{v}$ is the unit vector in the direction of the velocity of the plane and $\hat{h}$ is the unit vector in the direction from the ground to the plane.
This circulation forms a vortex line that leaves the wings and closes to the other wing by a loop behind the airplane. It is also this circulation that can be used to explain the lift in terms of magnus force on the wing as approximately $\rho v K$, where K is the circulation required by the correcting potential vortex.

6 Boundary Effects

Now, we have a one more final look at the effects at the boundaries with solids. Since we require that the fluid always stops at the boundary, the Navier-Stokes' equations close to boundary become

$$\left(\vec{v}\cdot\nabla + \frac{\partial}{\partial t}\right)\vec{v} = 0 = -\nabla P + \mu\nabla^2\vec{v} + \rho\vec{g}$$

For boundaries at the same potential level then we have a Poiseuille flow at the boundary

$$\mu\nabla^2\vec{v} = \nabla P$$

Since there is no flow into the boundary and there cannot be a continuum of stagnation points on the boundary (due to mass conservation), the velocity near the boundary is parallel to the boundary. Also, for a small piece of the boundary, the only characteristic direction is away from the boundary, hence the Laplacian operator becomes $\nabla^2 \to \frac{\partial^2}{\partial z^2}$, where z points away from the boundary.
This means that if the pressure is decreasing in the direction of motion of the fluid, the velocity magnitude profile in this direction is concavely increasing as travelling in the direction of z. If the fluid moves to a place with higher pressure, the profile is convexly increasing and can even be reverted close to the boundary. When the speed of the fluid close to the boundary is opposite than in the rest of the flow, we talk about the boundary layer separation. Then, there occurs a separation point, away from the boundary, where the velocity is zero for the second time.

6.1 Boundary Separation for Aerofoil

The air at the top of the wing is at lower pressure than the air at the bottom of the wing of a plane. This means that the air that travels on the top of the plane travels into the region of higher pressure, and therefore we could expect a boundary layer separation. But, no such separation is usually observed.
In modelling the plane, this requires the additional vortex behind the wing, such that it counteracts the boundary layer separation.

7 Drag Force

As we have seen, modelling the drag force (we tried it for a cylinder) is rather difficult as it usually involves a turbulence. We can approximate our solution for the cylinder as the same integral but only integrate over the front half of the cylinder (in sense of the flow), assuming that the pressure behind the cylinder is approximately constant at P_C and the flow has no clear overall direction. The overall pressure force on the cylinder is then in the direction of movement of the flow. From Bernoulli equation

$$P_0 - P_C \propto \frac{1}{2}\rho v^2$$

as the fluid behind the cylinder does not move very much. So, the overall force on the cylinder is

$$F \propto (P_0 - P_C)S \propto \frac{1}{2}\rho v^2 S$$

where S is the cross-section of the cylinder. We usually then state directly

$$F = \frac{1}{2}CS\rho v^2$$

where C is some constant. For reasonable velocities, this is a surprisingly good model.
Important idea is that the turbulence occurs due to the boundary separation (as boundary separation creates small vortices). Therefore, if we can reduce the boundary layer separation or the pressure gradient, we can delay when the separation occurs, and thus reduce the drag on an object. This is the key idea behind designing aero/hydrodynamic objects.

7.1 Drag crisis

We should notice that as $S \approx d^2$ and ρ and μ are material constants, we can say

$$F = \frac{1}{2}CS\rho v^2 = \frac{C\mu^2}{2\rho}\frac{\rho^2 v^2 d^2}{\mu^2} = bRe^2$$

Sometimes, the drag coefficient or the coefficient b is therefore plotted as a function of the Reynolds number squared.
Since the boundary layer separation occurs only for some sufficient flow velocity, there is a different value of the coefficient before this separation occurs, causing much more turbulent behaviour. But, usually change to turbulent behaviour momentarily decreases the drag coefficient, causing a drop in the drag force that is then regained as the object accelerates.