PX120: Electricity and Magnetism

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Disclaimer

This revision guide has been put together using the sweat, blood and tears of MathsPhys Society members. **It is intended as a revision tool and as such does not contain the entirety of the module, only the key elements. It was not complied, nor ratified by the University and most certainly not collated by any past, current or future lecturer of the afore mentioned course. As such it may be incomplete, may contain slight inconsistencies and although the utmost care has gone into ensuring they are kept minimal, slight inaccuracies.** Should you find any of the above mentioned please accept our apologies and find the time to drop us an email at the following address so we can change it in future editions, exec@warwickmathsphys.co.uk. First edition written and typset by Ben Newell in February 2009. Based off notes taken from the lectures given in 2007.

1 General Introduction

Despite being one of THE biggest first year physics courses, Electricity and Magnetism really doesn't have a lot to it. Many of the formulae will be familiar from A-Level Physics, particularly those involving capacitors, DC circuits and general expressions of $\mathbf{E}$ and $\mathbf{B}$. What the course really does is to take these ideas and places them into the mathematically rigorous framework of vector fields, and this is where the course can appear daunting. The sheer amount of closed integrals of vectors across surfaces is mind-boggling to start with, but what I hope to demonstrate is that these become immensely simplified when symmetry considerations are taken into account. When I took this course in 2007/8 Tony Arber gave 29 lectures, each self contained. These notes are based off my own revision notes, where each lecture is condensed into a few lines under it's own clear heading. Hopefully this will help you find what you're looking for.

2 Electricity

2.1 Charge and Coulomb's Law

Force of charge q_1 on q_2 is given by

$$\mathbf{F_{1,2}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the line joining q_1 to q_2. If $q_1, q_2 > 0$ this force is repulsive, and if $q_1, q_2 < 0$ this force is attractive.
The total force on q due to several charges q_i is simply the vector sum of the forces between q and each of the q_i.

2.2 Electric Fields

The electric field is defined as the force per unit charge, or in equation form

$$\mathbf{E} = \frac{\mathbf{F}}{q_0}$$

where q_0 is a "test charge", one which doesn't disturb the system.

The electric field due to a point charge q is given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector radially outwards from q.

Electric field lines are imaginary lines in space, tangent to $\mathbf{E}$, whose density is proportional to $|\mathbf{E}|$.
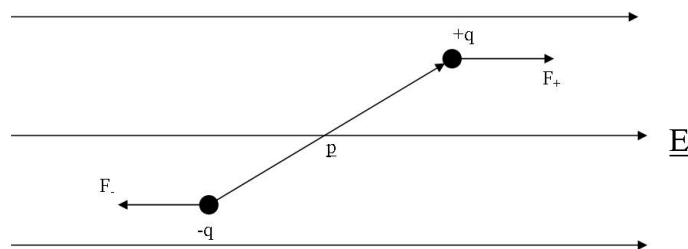
2.3 Dipoles

A dipole is a pair of charges, $+q$ and $-q$, separated by a fixed distance d.

The electric dipole moment is given by the following equation

$$\mathbf{p} = q\mathbf{d}$$

where $\mathbf{d}$ points from $-q$ to $+q$ and has magnitude d.

An electric dipole experiences no net force in an external electric field, $F_- = F_+$.

However in such an external field the dipole will move to align itself with the field, and hence experience a torque

$$\tau = \mathbf{p} \times \mathbf{E}$$

The potential energy of a dipole in an external field $\mathbf{E}$ is given by

$$U = -\mathbf{p} \cdot \mathbf{E}$$

2.4 Electric Flux

Electric flux is a measure of flow of electric field lines through a surface. For a uniform $\mathbf{E}$ flowing through a flat surface A, the electric flux is given by

$$\Phi_E = \mathbf{E} \cdot \mathbf{A}$$

where the direction of $\mathbf{A}$ is normal to the surface A.

For general $\mathbf{E}$ and $\mathbf{A}$ however, the electric flux is defined by what is known as a "Flux Integral"

$$\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$$

2.5 Gauss's Law

We now meet the first important equation in Electricity and Magnetism - Gauss's Law. This theoretically allows us to compute the electric field due to any given charge distribution by relating the electric flux through a closed surface to the charge enclosed by that surface. It turns out that this is only practical in highly symmetric situations, but luckily that's all that you'll ever be faced with in this course. So without any further ado, here it is...

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{encl}}{\epsilon_0}$$

Note that here $d\mathbf{A}$ points in the direction of the outward normal to the surface.

2.6 Applications of Gauss's Law

We'll go through an example of Gauss's Law, hopefully illustrating how all the symmetry arguments work and how the nasty surface integral just ends up being the magnitude of the field multiplied by an area (mostly). As a lecturer once said, an example is worth a thousand theorems.

Example: Finding the electric field due to an infinite plane of charge
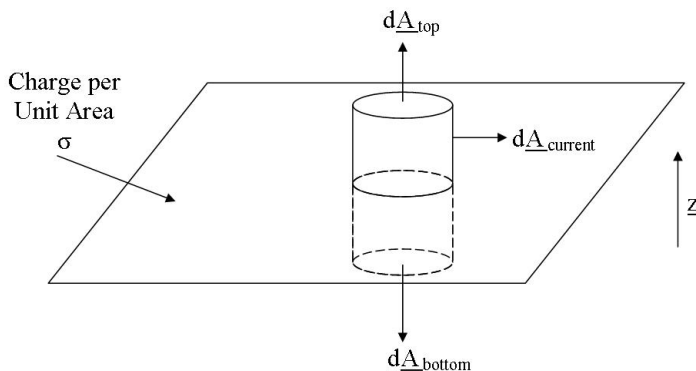
Firstly, we need to define the charge per unit area of the sheet or else we'll get nowhere, so let's call this σ. Before we get Gauss's law out, lets examine the set up and look for symmetries to make our lives easier. Firstly, since we're assuming the sheet to be infinite we can ignore any difficult edge effects and say that **E** must be entirely normal to the plane. Hence $\mathbf{E} = E\hat{\mathbf{z}}$ above the plane and $E(-\hat{\mathbf{z}})$ below the plane. We can also see fairly easily that E must be constant at a fixed height above (or below) the plane due to the symmetry of our infinite sheet - no matter where we are on the sheet, at constant height it looks the same. We're now ready to start using Gauss's Law. We'll use a cylinder of cross sectional area A passing through the sheet as our Gaussian surface (see diagram). This surface can be split into 3 sections - the top, the bottom and the curved surface - so we can write Gauss's Law as

$$\oint \mathbf{E} \cdot d\mathbf{A} = \int_{top} \mathbf{E} \cdot d\mathbf{A} + \int_{bottom} \mathbf{E} \cdot d\mathbf{A} + \int_{curved} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{encl}}{\epsilon_0}$$

Let's try and simplify this a bit. We've already said that **E** is entirely in the **z** direction, so we can see that on the curved surface $\mathbf{E} \cdot d\mathbf{A} = 0$ since $d\mathbf{A}$ is perpendicular to $\hat{\mathbf{z}}$ everywhere on the curved surface. Similarly on the top and bottom $d\mathbf{A}$ is parallel to **E** so $\mathbf{E} \cdot d\mathbf{A} = EdA$. We can also take E out of both integrals, since all points on the top and bottom surfaces are at a fixed height. So we can write

$$\oint \mathbf{E} \cdot d\mathbf{A} = E \int_{top} dA + E \int_{bottom} dA = \frac{Q_{encl}}{\epsilon_0}$$

To finish off, note that $Q_{encl} = \sigma A$ and that $\int_{surface} dA$ = area of surface. Then we get

$$2EA = \frac{\sigma A}{\epsilon_0} \Rightarrow \mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

above the plate, and the same thing in the negative **z** direction below the plate.

2.7   Charge on Conductors

Inside a conductor, $\mathbf{E} = 0$. All the excess charges are found on the surface of the conductor.

This leads us nicely to another application of Gauss's Law - finding the electric field above the surface of a conducting plate, charge per unit area σ. This is a good one to practice, since it uses the same method as we used in section 2.6 only this time instead of a thin plane our plate has a thickness. Try it! [1]

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[1]Hint: use a cylinder as before, only this time have the bottom surface *inside* the plate

2.8 Electric potential

The general formula for work done against a force $\mathbf{F}$ in moving from a to b is

$$W_{a \to b} = \int_a^b \mathbf{F} \cdot d\mathbf{l}$$

For a conservative force, this reduces nicely to

$$W_{a \to b} = -(U_b - U_a) = -\Delta U$$

We define the electric potential energy at a distance r from charge q as

$$U = \frac{q \, q_0}{4\pi\epsilon_0 r}$$

Since this is a scalar quantity, the potential energy of many charges is simply the algebraic sum of the individual potential energies.

The electric potential V is defined as

$$V = \frac{U}{q_0}$$

where U here is the electric potential energy of a point charge q_0.

Hence the electric potential of a point charge q is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

As a convention, we often define $V_\infty = 0$.

2.9 Calculating potentials

We define

$$V_{ab} = V_a - V_b = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Let's try an example. We'll try to calculate the potential at a distance r from an infinitely long line of charge, with charge per unit length λ. We can use Gauss's law [2] to show that the electric field due to this arrangement is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}$$

So clearly it's going to make sense to take $d\mathbf{l} = dr\hat{\mathbf{r}}$. We'll also need to define a reference distance, r_0, such that $V(r_0) = 0$. Then we can write

$$V(r) = V(r) - V(r_0) = \int_r^{r_0} \mathbf{E} \cdot d\mathbf{l} = \int_r^{r_0} \frac{\lambda}{2\pi\epsilon_0 r'} dr' = \frac{\lambda}{2\pi\epsilon_0} \int_r^{r_0} \frac{1}{r'} dr' = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

where we've used r' as a dummy variable since r is a limit of integration.

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[2]Take the surface to be a cylinder with axis concentric to the line of charge - give it a go!

2.10 Potential gradients

Somewhat stating the obvious, equipotential surfaces are surfaces at a constant potential. We can use this fact to derive that on an equipotential surface the electric field is perpendicular to the surface. On an equipotential surface, $V_{ab} = V_a - V_b = 0$ by definition. However we know $V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$ along any path between a and b, and hence $\mathbf{E} \cdot d\mathbf{l} = 0$. Then $\mathbf{E}$ is perpendicular to $d\mathbf{l}$ and since $d\mathbf{l}$ runs along the surface $\mathbf{E}$ must be perpendicular to the equipotential surface.

A conductor is an example of an equipotential volume.

NOTE: Equipotential lines are NORMAL to the electric field lines!

2.11 Capacitors

A bit of A-level revision here, learn (or remember) and apply! Standard capacitance equation:

$$C = Q/V$$

For a parallel plate capacitor:

$$C = \frac{\epsilon_0 A}{d}$$

where A is the area of the plate of the capacitor and d is the distance between the plates.

Remember that capacitors act opposite to resistors in circuits. For n capacitors in parallel:

$$C_T = \sum_{k=1}^{n} C_k = C_1 + C_2 + C_3 + ... + C_n$$

and in series:

$$\frac{1}{C_T} = \sum_{k=1}^{n} \frac{1}{C_k} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_n}$$

2.12 Field Energy

More equation-y goodness!

The potential energy of n point charges is given by

$$U = \sum_{i=1}^{n} \frac{1}{2} q_i \phi_i$$

Where ϕ_i is the potential at q_i due to all of the other point charges.

This allows us to work out the energy stored in a capacitor. We consider the two plates, one of charge Q and the other of charge $-Q$, and the potential difference, V, between them and apply the equation above. Then the energy stored in a capacitor is given by

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

From this we can work out the energy density of the $\mathbf{E}$ field between the plates, since this is where all the energy is stored. To do this we divide the energy by the volume between the plates and mess around algebraically to find the energy density of an $\mathbf{E}$ field is

$$u = \frac{1}{2}\epsilon_0 E^2$$

2.13 Dielectrics

Applying an **E** field to a dielectric moves the centre of electron charge distribution in the dielectric, effectively creating a dipole between the positive and negative charge centres. From the diagram, it is easy to see that all net charge lies on the surface of the dielectric. Dielectrics are incredibly useful in capacitors, as we will now see.

Since we haven't had enough equations lately, here are a few more!

Relative permittivity, ϵ_r, is given by

$$\epsilon_r = \frac{C}{C_0}$$

where C_0 is the initial capacitance of the capacitor and C is the capacitance when a dielectric is inserted between the plates of the capacitor. This can be rearranged to show that

$$C = \frac{\epsilon_r \epsilon_0 A}{d}$$

How does inserting a dielectric between the plates of a capacitor change V, E and U? All the following results can be derived using the above 2 equations and the relations we learnt in Section 2.11.

$$V = \frac{V_0}{\epsilon_r}$$

$$U = \frac{1}{2}\epsilon_0 \epsilon_r E^2$$

$$E = \frac{E_0}{\epsilon_r} = \frac{\sigma - \sigma_i}{E_0}$$

where σ is the charge density on the plate of the capacitor and σ_i is the charge density on the surface of the dielectric.

2.14 Current, Resistance and EMF

Lots more A-Level revision to be found here, $V = IR$ and $P = VI = I^2 R = \frac{V^2}{R}$ will certainly be familiar.

An important thing to note is that a force needs to be provided to keep the charges moving around the circuit, and this must be equal to $-q\mathbf{E}$.

The electromotive force (EMF) of a battery is the voltage across is when no current is flowing, and is defined as the work done by the force of the battery per unit charge. The name is misleading, since this isn't a force at all! We also need to remember that batteries have internal resistance, reducing the output voltage, which means we actually find

$$V = \epsilon - Ir$$

where r is the internal resistance of the battery.

2.15 DC Circuits

We conclude electricity with a look at Kirchhoff's laws and basic DC circuit theory.

Kirchhoff's First Law: There is no net current at a junction. $\sum I_i = 0$
Kirchhoff's Second Law: Around a closed loop, the sum of the EMF's is equal to the sum of the voltage drops.

A useful technique when solving problems involving Kirchhoff's second law is to use Maxwell current loops, where each loop is assigned it's own current. For example consider this set up:
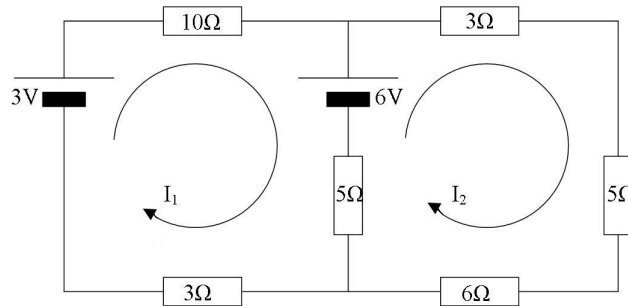
Here we have assigned a current to each loop in the circuit. What we do next is apply Kirchhoff's 2nd Law to each loop and solve the simultaneous equations. For the left hand loop

$$3 - 6 = -3 = 10I_1 + 5(I_1 - I_2) + 3I_1 = 18I_1 - 5I_2$$

and for the right hand loop

$$6 = 19I_2 - 5I_1$$

We solve these equations to find that $I_1 = \frac{87}{65}$ and $I_2 = \frac{165}{247}$. All we need now is the current flowing through the middle section of the circuit, which will be the difference between the two in the direction of the larger one. In our case the current in the centre will be $\frac{828}{1235}$ downwards, since $I_1 > I_2$.

Phew! That's electricity covered for now, we can forget all about it for a while as we build up a theoretical framework for magnetism. Sounds fun, doesn't it?

3 Magnetism

3.1 Magnetic Fields

Magnetic fields are generated by *moving charges*. This is a rather important difference between electric and magnetic fields (always useful to know in case of exam questions!).

The force on a charge q moving at velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Combining this with our earlier expression for the force on a charge in an electric field, we can write a general expression for a charge moving in both an electric and a magnetic field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This is the Lorentz Force Law, another exam favourite.

We define magnetic flux, Φ_B, similarly to how we defined electric flux:

$$\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$$

This leads us nicely onto another important distinction between electricity and magnetism, the fact that there are no magnetic monopoles. There is no 'charge' of magnetism, the picture below simply cannot happen, which means that magnetic field lines are always closed.
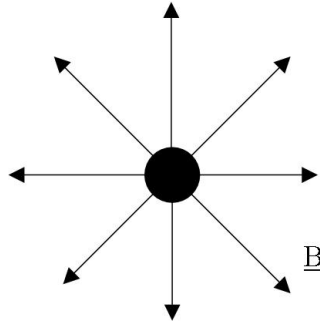
We can express this fact mathematically by the statement

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

Which simply means that magnetic flux going out of a closed surface is equal to magnetic flux coming back in through that surface. This is another Maxwell Equation, known as the Arber Equation[3] (just don't write that in an exam...)

Finally, we demonstrate the fact that a constant magnetic field does no work. There are 2 ways to do this, the easy way and the hard(er) way. We can either argue by using Newton's 2nd Law to show:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}$$

$$\Rightarrow m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = q\mathbf{v} \cdot (\mathbf{v} \cdot \mathbf{B}) = 0$$

Since $\mathbf{v} \cdot \mathbf{B}$ is perpendicular to $\mathbf{v}$ by definition, and the dot product of perpendicular vectors is zero. Finally we rewrite the left hand side to complete the proof

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2) = 0$$

This statement shows that the kinetic energy of the charge doesn't change in time, and hence no work is done on it.

We can prove the fact much easier by simply noting the definition of work as

$$W = \int \mathbf{F} \cdot d\mathbf{l}$$

and since $\mathbf{F}$ is perpendicular to $\mathbf{v}$ (and hence $d\mathbf{l}$),

$$\mathbf{F} \cdot d\mathbf{l} = 0 \Rightarrow W = 0$$

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[3]Ok, not really, but he'd like to think so and I'm not going to stop him!

3.2 Force on Current Loops

We can rewrite $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ as $\mathbf{F} = I\mathbf{l} \times \mathbf{B}$ using the relation between current, charge and time. From this we can deduce the force on a length $d\mathbf{l}$ of a wire carrying a current I to be

$$d\mathbf{F} = Id\mathbf{l} \times \mathbf{B}$$

We can also define the torque on a current loop in a magnetic field as $\tau = \mu \times \mathbf{B}$ where μ is the magnetic dipole moment and is defined as

$$\mu = NIA\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane of the current loop.

3.3 Biot-Savart Law

The Biot-Savart Law is a very useful formula for finding the magnetic field in certain situations. Without further ado...

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \mathbf{r}}{r^3}$$

or equivalently

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{Id\mathbf{l} \times \mathbf{r}}{r^3}$$

This is useful in calculating the magnetic field due to an infinitely straight wire, and as it happens not much else at all. Unfortunately this process is inherently stupid and tedious, so we'll just ignore it for now and move on to something all together more useful and fun.

3.4 Ampere's Law

Whereas the Biot-Savart Law generally leads to pain, suffering and misery when trying to find an expression for a general magnetic field, Ampere's Law brings mirth and sunshine to people everywhere. It relates the magnetic field to the current passing through a closed loop:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

Where I is the current passing through the loop being integrated over. This loop is known as an Amperean loop or path. This equation brings with it a sign convention for I. We define I to be positive if it runs in the same direction as the direction defined by the right hand rule[4].

This, unlike the Biot-Savart Law, gives us the expression for the magnetic field due to an infinitely long straight wire in just a few lines. We take the Amperean path to be a circle, centre concentric with the axis of the wire, and argue that by symmetry the magnetic field is constant at fixed radius and must be entirely angular by the right hand rule (thumb points in direction of current, fingers give direction of magnetic field). Hence $d\mathbf{l} = dl\hat{\theta}$ and $\mathbf{B} = B_\theta(r)\hat{\theta}$. So we get

$$\oint B_\theta dl = \mu_0 I$$

$$\Rightarrow B_\theta \oint dl = \mu_0 I$$

$$\Rightarrow B_\theta 2\pi r = \mu_0 I$$

$$\Rightarrow B_\theta(r) = \frac{\mu_0 I}{2\pi r}$$

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[4]See front cover: the fingers curl around in direction of path, thumb points in positive direction. See, it's not *just* a sexual innuendo.

3.5   Applications of Ampere's Law

Or, rather, application since we're only going to do the one, but it's a big one. We're going to find the electric field everywhere due to an infinitely long solenoid (infinitely long means we can ignore irritating end effects) with radius a. We'll assume that the wire has been wound both forwards and backwards along the solenoid so that there's no net current in the $\mathbf{z}$ direction. We'll also assume number of turns per unit length, n, is large, so we can assume axial symmetry. For complete generality, we'll write $\mathbf{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\theta} + B_z\hat{\mathbf{z}}$.
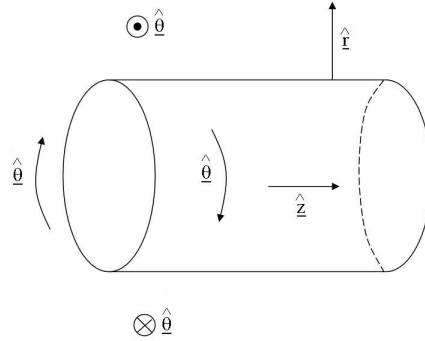

Let's start with the θ component. Take the Amperean path to be a circle with its centre concentric with the axis of the solenoid. Whether the radius of the circle is larger or smaller than the radius of the solenoid is irrelevant, since we'll always have $I_{encl} = 0$ as there is no net current in the $\mathbf{z}$ direction. On this path, $d\mathbf{l} = rd\theta\hat{\theta}$. Hence by Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint B_\theta rd\theta = B_\theta 2\pi r = 0 \Rightarrow B_\theta = 0$$

So there's no angular component to the magnetic field. Let's move on to the radial component. We'll actually use Arber's Law to show that this is equal to zero instead of Ampere's Law. Take a cylinder of length L, axis concentric with the axis of the solenoid, and apply Arber's Law. Note that the contributions from the ends of the cylinder will cancel each other out. Then Arber's Law tells us

$$\int_{curved} \mathbf{B} \cdot d\mathbf{A} = \int_{curved} B_r dA = B_r 2\pi L = 0 \Rightarrow B_r = 0$$

since on the curved surface $d\mathbf{A} = dA\hat{\mathbf{r}}$. So we've got no radial component to our magnetic field anywhere either, which leaves us with just the $\mathbf{z}$ component to consider. For this we need to consider 2 cases, one for the outside and one for the inside. We'll look at the outside first. Take an Amperean path of a rectangle, as in this picture.
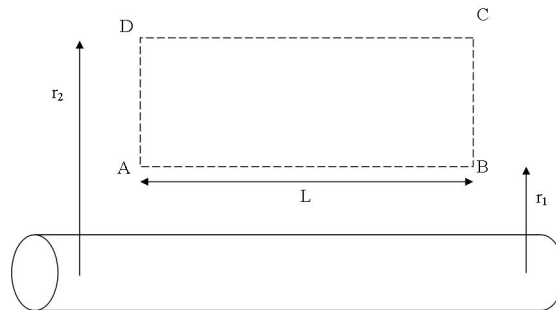
Note that contributions from AD and CB will be zero, since along these $d\mathbf{l} = dr\hat{\mathbf{r}}$ and $B_r = 0$. So that leaves us with sides AB and CD.

$$\text{On AB: } \int \mathbf{B} \cdot d\mathbf{l} = B_z(r_1)L$$

$$\text{On CD: } \int \mathbf{B} \cdot d\mathbf{l} = -B_z(r_2)L$$

Now note that there is no current enclosed by this rectangle since it is completely outside the solenoid. Let's apply Ampere's Law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_z(r_1)L - B_z(r_2)L = \mu_0 I_{encl} = 0 \Rightarrow B_z(r_1) = B_z(r_2)$$

Send $r_2 \to \infty$. Then we expect $\mathbf{B} \to 0$, so $B_z(r_1) = B_z(r_2) = 0 \Rightarrow B_z(r) = 0$ if $r > a$.

You may be thinking this is a pretty stupid example, since so far there have been no magnetic fields. However, we've got one left to consider, B_z inside the solenoid. For this, we again take an Amperean path of a rectangle, only this time it intercepts the cylinder as below.
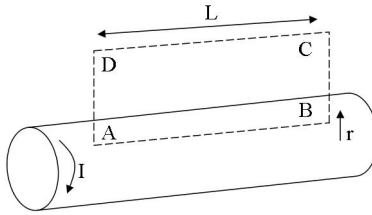
From what we've learnt so far there can only be one contribution to $\mathbf{B} \cdot d\mathbf{l}$ - that along AB. Also note that we have an enclosed current, $I_{encl} = nLI$, where n is the number of turns per unit length along the solenoid. Let's apply Ampere's Law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int B_z(r)dl = B_z(r)L = \mu_0 I_{encl} = \mu_0 nLI$$

$$\Rightarrow B_z(r) = \mu_0 nI \text{ for } r < a$$

So in conclusion we've shown that outside the solenoid, $\mathbf{B} = 0$ and inside the solenoid $\mathbf{B} = \mu_0 nI\hat{\mathbf{z}}$. Phew!

3.6 Magnetic Materials

Warning: This chapter is boring (yes, more boring than the others). It's also probably not likely to be on the exam, since it's just a bunch of facts, but who knows how things have changed since I did the exam, so don't quote me on that one.

In magnetic materials, atoms behave like magnetic dipoles. The orbiting electrons act like a current, causing many tiny magnetic dipoles to arise. These magnetic dipoles are either induced by an external applied magnetic field or are permanent. It's easy to see that these little dipoles give rise to a surface current on the material. If we place one of these materials into a magnetic field $\mathbf{B}_0$ then one of two things will happen. Either magnetic dipoles will be induced in the material, or the permanent magnetic dipoles will align with the field (as we saw in section 3.2). These magnetic dipoles change the magnetic field inside the material according to

$$\mathbf{B} = \mu_r \mathbf{B}_0$$

where μ_r is the relative permeability of the material, and is a property of the material.

There are 4 types of magnetic material.

Diamagnetic: $\mu_r < 1$ so **B** is reduced. No permanent magnetic dipoles, induced dipoles are antiparallel to the applied magnetic field by Lenz's Law. Typically, $0.99990 < \mu_r < 0.99999$
Paramagnetic: $\mu_r > 1$ so **B** is increased. Permanent atomic magnetic dipoles which are randomly oriented until external field is applied. Typically, $1.00001 < \mu_r < 1.003$
**Ferromagnetic:** $\mu_r >> 1$ which means a greatly increased magnetic field inside the material. μ_r is of the order of 10^3 and is not constant but instead varies with $\mathbf{B}_0$.
Hard Ferromagnets: These are permanent magnets, which maintain **B** even when $\mathbf{B}_0 = 0$. Hence we cannot define μ_r for a hard ferromagnet.

4   Electromagnetism & AC Circuits

4.1   Faraday's Law

Faraday's Law states that *"The induced EMF in a closed loop is equal to minus the rate of change of magnetic flux through the loop"*. I feel it's best represented (and remembered) by this wonderful diagram:
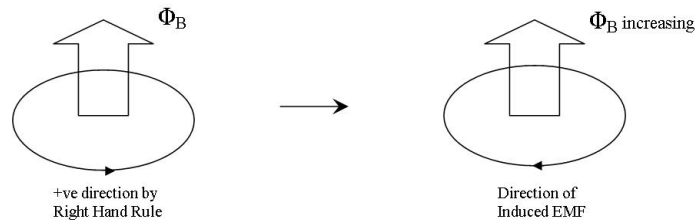
Where the large arrow indicates that Φ_B is increasing in this direction.

Mathematically, Faradays Law can be expressed

$$\epsilon = -\frac{d\Phi_B}{dt}$$

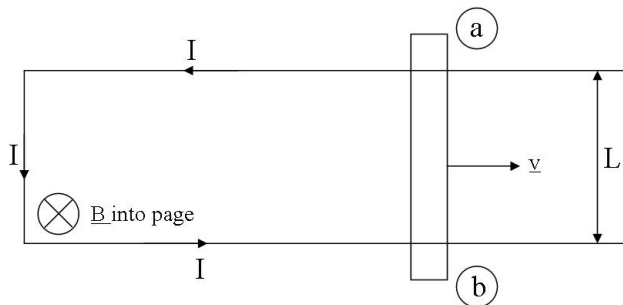for a loop consisting of a single turn of wire and

$$\epsilon = -N\frac{d\Phi_B}{dt}$$

for a loop consisting of N turns.

4.2   Lenz's Law

Lenz's Law states that *"The effect of the induced EMF is such that it opposes the changing flux which produced it"*. It is a statement of energy conservation.

Lenz's Law is best shown through an example:

The charges in the crossbar move in the **B** field subject to $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. This leads to a potential difference between a and b since positive charges are forced towards a. Hence an **E** field is established from a to b. This electric field exerts a force $\mathbf{F} = q\mathbf{E}$ on the charges in the crossbar in the *opposite direction* to the force from the magnetic field. Hence in equilibrium these two forces must be equal and hence we can say that $E = vB$ downwards. Since $E = \frac{V}{d}$, the voltage between a and b is $EL = vLB$. This is known as a motional EMF since our circuit is effectively "moving" through the **B** field as the crossbar moves to the right, hence increasing the amount of magnetic flux enclosed.

If we now consider the work done we can derive an expression for ϵ. In our example,

$$W = \oint \mathbf{F} \cdot d\mathbf{l} \Rightarrow \epsilon = \frac{1}{q} \oint \mathbf{F} \cdot d\mathbf{l} = \oint \left((\mathbf{v} \times \mathbf{B}) - \mathbf{E} \right) \cdot d\mathbf{l}$$

But since $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ always, the above equation implies

$$\epsilon = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

4.3   Induced Electric Field

Looking at Faraday's Law, it's easy to see that we get an induced EMF even if the circuit remains stationary and **B** changes. Let's look at an example of this phenomenon.

Consider a solenoid with n turns per unit length carrying a current I_s which has a loop of wire around it. The centre of the loop is concentric with the axis of the solenoid and has radius r and resistance R. As we saw in section 3.5, the solenoid gives rise to a magnetic field $B = \mu_0 n I_s$ inside the solenoid parallel to the axis of the solenoid. Hence $\Phi_B = \mu_0 n I_s A$ since **B** is parallel to $d\mathbf{A}$. Imagine now that I_s changes in time. Then Φ_B, since it depends on I_s will also vary and hence an EMF will be induced.

$$\epsilon = -\frac{d\Phi_b}{dt} = -\mu_0 n A \frac{dI_s}{dt}$$

$$\Rightarrow I_w = \frac{\mu_0 n A}{R} \frac{dI_s}{dt}$$

Where I_w is the current in the wire and the second line follows from $V = IR$. Note that the minus sign in our expression for the EMF tells us the direction of the EMF, hence why I've omitted it in the expression for I_w.

This may seem all well and good, but actually masks a REALLY REALLY BIG PROBLEM. As we know, $\epsilon = \oint ((\mathbf{v} \times \mathbf{B}) - \mathbf{E}) \cdot d\mathbf{l}$. In the example of the moving wire everything was cool, since our moving wire gave rise to a magnetic force $\mathbf{F} = \mathbf{v} \times \mathbf{B}$. But for our loop of wire *nothing*

is moving. So $\mathbf{F} = \mathbf{v} \times \mathbf{B} = 0$ since $v = 0$. You may think "Ok, fair enough, this leaves us with $\epsilon = \oint \mathbf{E} \cdot d\mathbf{l}$", which is correct, *but we know* $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ *always.* So we have a non zero EMF being equal to zero. Clearly something is wrong here. To rectify this problem, we need that

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_b}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}$$

which seems pretty ridiculous given what we've learnt so far about electric fields. To reconcile this with what we've seen so far, Faraday suggested that a changing magnetic flux generates *closed loops of* $\mathbf{E}$. These loops are simply created in space by the changing flux and fit in with what we've come to expect since no net work will be done around a closed loop. We can't actually define a potential for the induced electric field, and hence we bypass the problem completely. Awesome.

4.4 Inductors

As we've seen so far, a current generates a magnetic field giving rise to a magnetic flux. If this current, and hence flux, is changing in time then an EMF opposing the change in current is created. The ratio of the magnetic flux to the current is known as the (self) inductance of the circuit. In maths:

$$N\Phi_B = LI$$

Where L is the inductance. The unit of inductance is the Henry, H.

The EMF induced in an inductor is given by

$$\epsilon = -L\frac{dI}{dt}$$

Again, Lenz's law tells us that this EMF opposes the changing current.

Inductors can store energy in the magnetic field. This energy is given by the formula

$$U = \frac{1}{2}LI^2$$

and the energy density by

$$u = \frac{B^2}{2\mu_0}$$

4.5 AC Circuits

Firstly, a rant. For some ridiculous and morally reprehensible reason, physicists have inherited an engineering convention in denoting $\sqrt{-1}$ by the letter j, instead of the more common i, since engineers decided to use i for current until they realised that $\sqrt{-1}$ might actually be useful for something. I'm not going to do this, because it's stupid.

In an AC Circuit, the voltage (and hence pretty much everything else) varies in time. V is not just V anymore. We now have

$$V = V_0 \cos \omega t$$

where $\omega = 2\pi f$ and is the angular frequency of the circuit. However, no one really likes playing around with sines and cosines, so instead we use complex numbers for our AC voltage, current etc.

$$\tilde{V} = V_0 e^{i\omega t}$$

where the tilde is used to show we're talking about a complex quantity.

When we look at voltage and current as complex quantities we'll consider phase differences and will be looking at Argand diagrams. It's useful to remember that multiplying a complex quantity by $e^{i\phi}$ rotates our quantity by an angle ϕ anticlockwise about the origin on the Argand diagram.

4.6 Complex Impedance

In our new world of complex voltages and currents, we are forced to do away with our old favourite $V = IR$ and come to terms with something altogether new - complex impedance. Impedance is a lot like resistance, but contains extra information about phase shifts. $V = IR$ becomes

$$\tilde{V} = \tilde{I}\tilde{Z}$$

which, conveniently, isn't too dissimilar. In order to analyse AC circuits, we'll need expressions for the complex impedance in a resistor, capacitor and an inductor.

In a resistor, $V = IR$, so is $V = V_0 \cos \omega t$ then $I = \frac{V_0}{R} \cos \omega t$. So in a resistor V and I are in phase, and $\tilde{Z}_R = R$.

In a capacitor, we know $Q = CV \Rightarrow \frac{dQ}{dt} = I = C\frac{dV}{dt}$. Using our complex notation this means

$$\tilde{I} = C\frac{d}{dt}(V_0 e^{i\omega t}) = i\omega C V_0 e^{i\omega t} = i\omega C \tilde{V}$$

Hence from $\tilde{V} = \tilde{I}\tilde{Z}$ we have that $\tilde{Z}_C = \frac{1}{i\omega C}$. Also, since $i = e^{i\frac{\pi}{2}}$ we can see that in a capacitor the current *leads* the voltage by $\frac{\pi}{2}$.

Finally, in an inductor we have that $V = L\frac{dI}{dt}$. Assuming $\tilde{I} = I_0 e^{i\omega t}$ and switching to complex notation we find

$$\tilde{V} = L\frac{d}{dt}(I_0 e^{i\omega t}) = i\omega L \tilde{I}$$

So using $\tilde{V} = \tilde{I}\tilde{Z}$ again, we can see that $\tilde{Z}_L = i\omega L$. Hence the current *lags* the voltage by $\frac{\pi}{2}$ in an inductor.

For convenience, or in case you didn't want to read all that, here's a quick reference!

$$\text{Resistor: } \tilde{Z}_R = R \Rightarrow \text{voltage and current in phase}$$

$$\text{Capacitor: } \tilde{Z}_C = \frac{1}{i\omega C} \Rightarrow \text{current leads voltage by } \frac{\pi}{2}$$

$$\text{Inductor: } \tilde{Z}_L = i\omega L \Rightarrow \text{voltage leads current by } \frac{\pi}{2}$$

A few words of warning/facts that could be asked for in exams: Kirchhoff's Laws apply to an AC circuit only if the physical size of the circuit is much much less than the wavelength of the AC supply and the transit time of any electromagnetic radiation is much much less than the period of the AC supply.

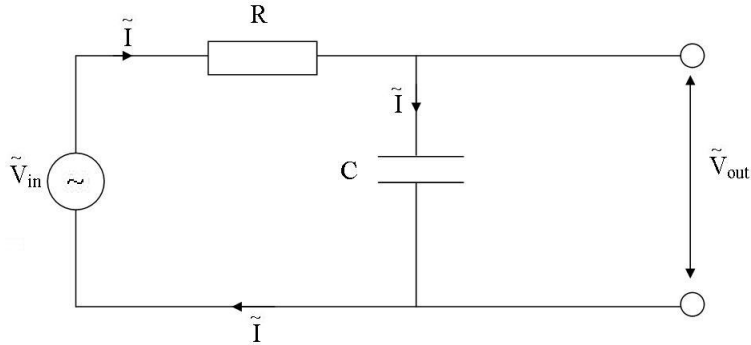When analysing AC circuits, complex impedances behave like resistors mathematically, ie:

$$\tilde{Z}_{total} = \tilde{Z}_1 + \tilde{Z}_2 + ... \text{ for components in series}$$

$$\frac{1}{\tilde{Z}_{total}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + ... \text{ for components in parallel}$$

4.7   LCR Circuits

Here are a few examples of AC circuits involving resistors, capacitors and inductors. In each example, we're aiming to find $\tilde{V}_{out}$.

Example 1

First we need the total complex impedance of our circuit, so we can derive an expression for $\tilde{I}$.

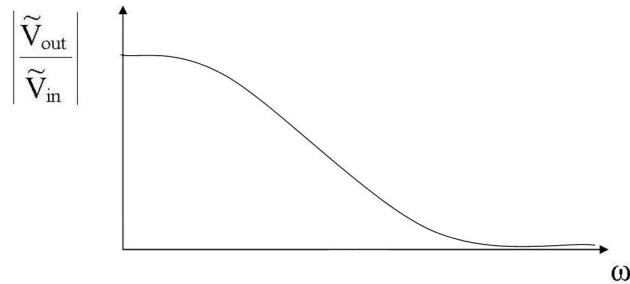$$\tilde{Z}_{total} = \tilde{Z}_R + \tilde{Z}_C = R + \frac{1}{i\omega C}$$

Hence, from $\tilde{V} = \tilde{I}\tilde{Z}$ we find
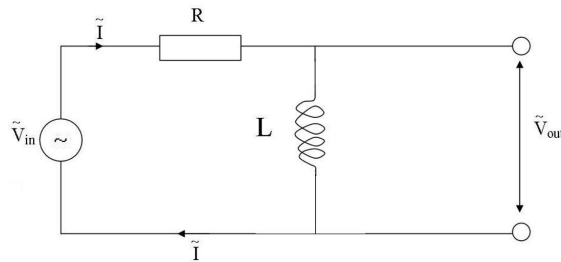
$$\tilde{I} = \frac{\tilde{V}_{in}}{R + \frac{1}{i\omega C}}$$

Finally, we use that $\tilde{V}_{out} = $ voltage across capacitor $= \tilde{I}\tilde{Z}_C$ to get

$$\tilde{V}_{out} = \frac{\tilde{V}_{in}}{1 + i\omega CR}$$

Looking at the graph of $|\frac{\tilde{V}_{out}}{\tilde{V}_{in}}|$ against ω we can see that this circuit is a low pass filter (low frequencies give higher output voltages than higher ones).
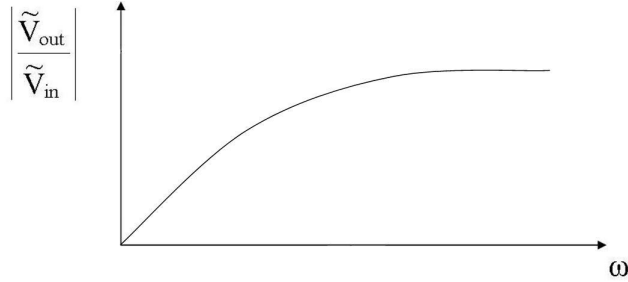

Example 2

Swapping the capacitor in the previous example for an inductor leads to the opposite effect. Now

$$\tilde{Z}_{total} = R + i\omega L \Rightarrow \tilde{I} = \frac{\tilde{V}_{in}}{R + i\omega L}$$
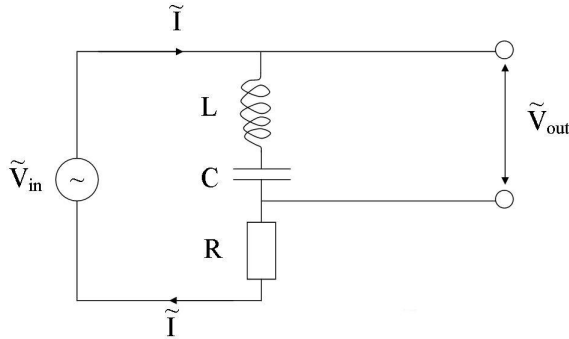
In this example, $\tilde{V}_{out} = $ voltage across inductor $ = \tilde{I}\tilde{Z}_L$ which gives

$$\tilde{V}_{out} = \frac{i\omega L \tilde{V}_{in}}{R + i\omega L}$$

From the graph of this function, it's easy to see that this circuit is a high pass filter.

Example 3

For our last example, we will stick in everything we've learnt about and see what happens.

$$\tilde{Z}_{total} = R + \frac{1}{i\omega C} + i\omega L \Rightarrow \tilde{I} = \frac{\tilde{V}_{in}}{R + i(\omega L - \frac{1}{\omega C})} \Rightarrow \tilde{V}_{out} = \frac{i(\omega L - \frac{1}{\omega C})\tilde{V}_{in}}{R + i(\omega L - \frac{1}{\omega C})}$$

This utterly ridiculous function gives us a rather nice graph, telling us that this circuit is a notch filter at $\omega = \frac{1}{LC}$. If we instead connected $\tilde{V}_{out}$ across the resistor, this would give us a band pass filter at the same frequency.
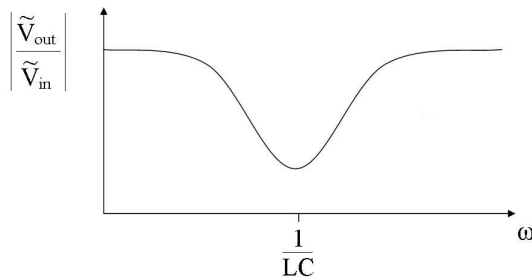
4.8   AC Power

In capacitors and inductors, voltage and current are $\frac{\pi}{2}$ out of phase. So we can write $\tilde{V} = V_0 \cos \omega t$ and $\tilde{I} = I_0 \sin \omega t$. If we now calculate the power dissipated in these components, we find

$$< P >=< VI >= I_0 V_0 < \sin(\omega t) \cos(\omega t) >= 0$$

So no power is dissipated in a capacitor or an inductor in an AC circuit. Pointy brackets mean "average value of", in case you didn't already know!

In a resistor however, voltage and current are in phase, so we get

$$< P >= I_0^2 R < \cos^2(\omega t) >= \frac{1}{2} I_0^2 R$$

So in an LCR circuit all power is dissipated in the resistor.

A couple of useful definitions:

$$I_{RMS} = \frac{I_0}{\sqrt{2}} \qquad V_{RMS} = \frac{V_0}{\sqrt{2}}$$

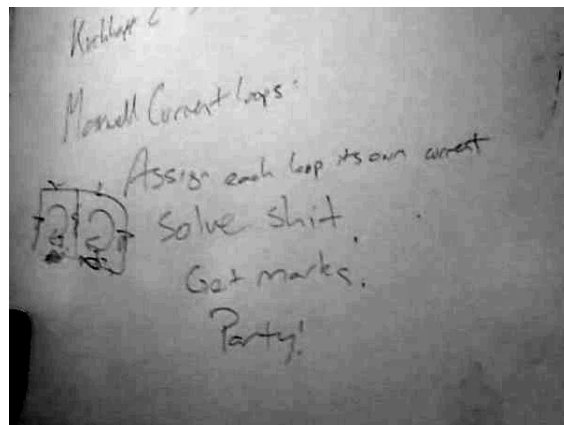Using these we can rewrite our definition of the power dissipated in a resistor.

$$< P >= I_{RMS}^2 R = I_{RMS} V_{RMS} = \frac{V_{RMS}^2}{R}$$

5   The End

And that's everything!

As for the exam, I can almost guarantee you WILL be asked to do at least one of the vector integrals so practice these throughly! Learn the symmetry arguments by heart as there will always be a few marks for knowing those. Also remember that there will be marks simply for knowing definitions of equations and concepts so don't forget to brush up on those too. Although the course seems large, once you start getting to grips with it the rest will fall into place easily.

Final words of encouragement: Electricity and Magnetism is easy! Seriously! Don't panic!

How to pass E&M